MAE 306 Notes #7c
Engineering Mathematics II

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1 How to Exploit Superposition

When the given problem (including the initial and boundary condtions) is strictly linear, the principle of superposition applies, and it should be fully exploited.

Consider the following PDE:

$$\frac{\partial^2 u}{\partial t^2} + \mathcal{L}(u) = S(\mathbf{x}, t) \tag{1}$$

where $\mathcal{L}$ is a self-adjoint second order differential operator. To be specific, we confine our attention to:

$$\mathcal{L}(u) = -\nabla^2 u \tag{2}$$

which, in Cartesian (x, y, z) coordinates, is:

$$\mathcal{L}(u) = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},\right) \tag{3}$$

In cylindrical polar (r, θ, z) coordinates, it is:

$$\mathcal{L}(u) = -\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2},\right) \tag{4}$$

To confine our attention further, we consider the one-dimensional (x, t) case so that $u(x, t)$ and $S(x, t)$. The domain of interest is $0 \leq x \leq \ell$ and $t \geq 0$.

The initial and boundary conditions are:

$$\begin{aligned} u(x,0) &= f(x), && (5) \\ \frac{du}{dt}(x,0) &= p(x), && (6) \\ u(0,t) &= L(t), && (7) \\ u(\ell,t) &= R(t). && (8) \end{aligned}$$

where $f(x)$, $p(x)$, $L(t)$ and $R(t)$ are known, differentiable (and you know we can cheat on this!) functions.

1.1 Splitting the initial conditions

We can split u into two parts:

$$u(x,t) = U_1(x,t) + U_2(x,t). \tag{9}$$

We choose to define U_1 and U_2 by:

$$\begin{aligned} \frac{\partial^2 U_1}{\partial t^2} + \mathcal{L}(U_1) &= S(x,t), && (10a) \\ U_1(x,0) &= f(x), && (10b) \\ \frac{\partial U_1}{\partial t}(x,0) &= 0, && (10c) \\ U_1(0,t) &= L(t), && (10d) \\ U_1(\ell,t) &= R(t). && (10e) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 U_2}{\partial t^2} + \mathcal{L}(U_2) &= 0, && (11a) \\ U_2(x,0) &= 0, && (11b) \\ \frac{\partial U_2}{\partial t}(x,0) &= p(x), && (11c) \\ U_2(0,t) &= 0, && (11d) \\ U_2(\ell,t) &= 0. && (11e) \end{aligned}$$

I assume we all know how to solve for U_2. Let us focus on how to solve $U_1(x,t)$.

We further split U_1 as follows:

$$U_1(x,t) = U_3(x,t) + U_4(x,t) \tag{12}$$

where U_3 carries the burden of being the unknown to be found, and

$$U_4(x,t) = L(t) + \frac{(R(t) - L(t))x}{\ell}. \tag{13}$$

Note that U_4 has been "cooked up" so that $U_4(0,t) = L(t)$ and $U_4(\ell,t) = R(t)$. We have $\mathcal{L}(U_4) = 0$, and

$$\frac{\partial^2 U_4}{\partial t^2} = \ddot{L}(t) + \frac{(\ddot{R}(t) - \ddot{L}(t))x}{\ell} = \ddot{U}_4(x,t). \tag{14}$$

The PDE and initial and boundary conditions for the new unknown U_3 are as follows:

$$\frac{\partial^2 U_3}{\partial t^2} + \mathcal{L}(U_3) = S_3(x,t), \tag{15a}$$

$$U_3(x,0) = f_3(x), \tag{15b}$$

$$\frac{\partial U_3}{\partial t}(x,0) = p_3(x), \tag{15c}$$

$$U_3(0,t) = 0, \tag{15d}$$

$$U_3(\ell,t) = 0. \tag{15e}$$

where

$$S_3(x,t) = S(x,t) - \left(\ddot{U}_4(x,t) + \mathcal{L}(U_4)\right), \tag{15f}$$

$$f_3(x) = f(x) - U_4(x,0), \tag{15g}$$

$$p_3(x) = -\dot{U}_4(x,0). \tag{15h}$$

Essentially, the problem for U_3 is almost the same as the original problem for u—except that the boundary conditions are now "homogeneous."

We further split U_3 into two parts:

$$U_3(x,t) = U_5(x,t) + U_6(x,t). \tag{16}$$

We let U_3 take care of $S_3(x,t)$ and $f_3(x)$, and let U_6 take care of $p_3(x)$—following the same ideas as before. We have:

$$\frac{\partial^2 U_5}{\partial t^2} + \mathcal{L}(U_5) = S_3(x,t), \tag{17a}$$

$$U_5(x,0) = f_3(x), \tag{17b}$$
$$\frac{\partial U_5}{\partial t}(x,0) = 0, \tag{17c}$$
$$U_5(0,t) = 0, \tag{17d}$$
$$U_5(\ell,t) = 0. \tag{17e}$$

and

$$\frac{\partial^2 U_6}{\partial t^2} + \mathcal{L}(U_6) = 0, \tag{18a}$$
$$U_6(x,0) = 0, \tag{18b}$$
$$\frac{\partial U_6}{\partial t}(x,0) = p_3(x), \tag{18c}$$
$$U_6(0,t) = 0, \tag{18d}$$
$$U_6(\ell,t) = 0. \tag{18e}$$

The problem for U_6 is generically the same as that for U_2. So we assume that you do solve for both U_2 and U_6—using separation of variables.

Now, we need only to focus on U_5.

1.2 Basis functions

Consider now just our operator $\mathcal{L}$. It is easily verified that $\mathcal{L}$ is self-adjoint. In addition, when both u and v are zeros at $x = 0$ and $x = \ell$, we have:

$$< \mathcal{L}(u), v >=< u, \mathcal{L}(v) > \tag{19}$$

because the term called "bilinear concommitant" is zero.[1]

We now go find the eigenfunctions ϕ_n of the operator $\mathcal{L}$. The eigenfunctions are defined by:

$$\mathcal{L}(\phi_n) = \kappa_n \phi_n, \quad n = 1, 2, \ldots. \tag{20}$$

where κ_n is the eigenvalue. The boundary conditions for the eigenfunctions are:

$$\phi_n(0) = 0, \quad \phi(\ell) = 0, \tag{21}$$

[1] I assume we all remember that $< u, v >$ denotes the inner product of u and v:

$$< u, v >= \int_0^\ell uv dx.$$

as required by (19). For the one-dimensional (x, t) case, we have:

$$\phi_n(x) = C_n \sin(\sqrt{\kappa_n} x) + D_2 \cos(\sqrt{\kappa_n} x). \tag{22}$$

As usual, one of the coefficients D_n and the eigenvalue κ_n are determined by the boundary conditions:

$$D_n = 0, \quad \sin(\sqrt{\kappa_n} \ell) = 0, \tag{23}$$

yielding:

$$\kappa_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, \ldots. \tag{24}$$

If we were dealing with the cylindrical polar case, the eigenfunctions would be Bessel functions. We shall normalize the ϕ_n's such that:

$$< \phi_n, \phi_n >= 1. \tag{25}$$

Of course, by the Sturm-Liouville theory we know the eigenfunctions are orthgonal to each other:

$$< \phi_m, \phi_n >= 0, \quad m \neq n. \tag{26}$$

Hence, we have just reinvented Fourier Sine Series.

1.3 Expansions of S_3 and f_3

We now expand $S_3(x, t)$ and $f_3(x, t)$ in terms of the eigenfunctions of $\mathcal{L}$:

$$S_3(x, t) = \sum_{n=1}^{\infty} A_n \phi_n(x), \tag{27}$$

$$f_3(x, t) = \sum_{n=1}^{\infty} B_n \phi_n(x), \tag{28}$$

where

$$A_n(t) = < S_3, \phi_n > . \tag{29}$$

$$B_n(t) = < f_3, \phi_n > . \tag{30}$$

In other words, $A_n(t)$ and $B_n(t)$ are known functions of t.

1.4 Expansion of U_5

Why not expand the desired unknown $U_5(x,t)$ also? Indeed, why not?

We have, formally:

$$U_5(x,t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x) \tag{31}$$

Substituting into (17a), we have:

$$\sum_{n=1}^{\infty}\left(\frac{1}{a^2}\ddot{T}_n\phi_n + \kappa_n T_n\phi_n\right) = \sum_{n=1}^{\infty} A_n\phi_n. \tag{32}$$

Taking the inner product of (32) with ϕ_m, and taking advantage of orthogonality of eigenfunctions of self-adjoint $\mathcal{L}$, we have:

$$\frac{1}{a^2}\ddot{T}_m + \kappa_m T_m = A_m(t), \quad m = 1, 2, \ldots, \tag{33}$$

which are second order ODEs for the $T_m(t)$'s. Remember, the $A_m(t)$'s are known functions of t. The initial conditions for the T_n's can easily be deduced from (17b) and (17c):

$$T_n(0) = B_n(0), \quad \dot{T}_n(0) = 0, \quad n = 1, 2, \ldots. \tag{34}$$

So, the solution for U_3 boils down to solving (33) for the $T_n(t)$'s which honor the initial conditions (34).

1.5 What Insights Did We Learn?

Equation (33) is the ODE for an harmonic oscillator (with no damping) under the influence of a forcing term $A_m(t)$.

What happens if some particular A_i is approximately (or precisely) equal to $\sin(a\sqrt{\kappa_i}t)$ or $\cos(a\sqrt{\kappa_i}t)$? Resonance! Pow!

If we had chosen the method of characteristics to handle this problem, we would be doing a lot more computation, graphing the characteristics and working with the characteristic relations. If you want to know what happened after when $t = 2$ years, you will be an old man/woman when you finish your calculation. And you would probably not recognize that you have a resonance issue to be concerned about until you reach middle age.