1 Reading and Homework Assignments

Study Chapter 20, Laplace Equations, pp. 1058-1107 of Michael Greenberg’s *Advanced Engineering Mathematics*, and do the problem assigned below.

The problems are due on Tuesday, April 6th, 1999, 5PM. Please submit your homework to the MAE 306 homework IN tray outside D-302, E.Q.

2 Supplementary, Readings and Problems

Again, Greenberg limits himself pretty much to separation of variables (§20.1, 20.2 and 20.3), with a brief section on Fourier Transform (§20.4), and a brief section on numerical methods (§20.5).

2.1 Curvilinear Coordinates

The vector definition of the Laplacian operator \( \nabla^2 \) is:

\[
\nabla^2 u \equiv \nabla \cdot \nabla u
\]

If you need the explicit formula for the Laplacian in some coordinate system, you just need to find out the formula for the gradient of a scalar, and the formula for the divergence of a vector.

Consider the curvilinear coordinate system \((x_1, x_2, x_3)\). The “length” or the “distance” between two adjacent points is given by:

\[
(ds)^2 = (h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2
\]
where \( h_1, h_2 \) and \( h_3 \) may be functions of \((x_1, x_2, x_3)\). Let \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \) be unit vectors in the \((x_1, x_2, x_3)\) directions, respectively. The following are useful formulas (Sokolnikoff and Redheffer, Mathematics of Physics and Modern Engineering, McGraw Hill; page 416.):

\[
\nabla \mathbf{u} = \mathbf{e}_1 \frac{\partial u}{\partial x_1} + \mathbf{e}_2 \frac{\partial u}{\partial x_2} + \mathbf{e}_3 \frac{\partial u}{\partial x_3},
\]

(3)

\[
\text{Div} \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial (V_1 h_2 h_3)}{\partial x_1} + \frac{\partial (V_2 h_3 h_1)}{\partial x_2} + \frac{\partial (V_3 h_1 h_2)}{\partial x_3} \right),
\]

(4)

\[
\text{Curl} \mathbf{V} = \nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left| \begin{array}{ccc}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
h_1 V_1 & h_2 V_2 & h_3 V_3
\end{array} \right|,
\]

(5)

\[
\nabla^2 u = \nabla \cdot \nabla u = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial u}{\partial x_3} \right) \right],
\]

(6)

Examples are:

**Cartesian** \((x, y, z)\): \(x_1 = x, x_2 = y, x_3 = z; h_1 = h_2 = h_3 = 1\).

**Cylindrical Polar** \((r, \theta, z)\): \(r\) is distance from the \(z\) axis. \(x_1 = r, x_2 = \theta, x_3 = z; h_1 = 1, h_2 = r = x_1, h_3 = 1\).

**Spherical** \((\rho, \phi, \theta)\): \(\rho\) is distance from the origin, \(\phi\) is angle from the polar axis, and \(\theta\) is azimuthal angle about the polar axis. \(x_1 = \rho, x_2 = \phi, x_3 = \theta; h_1 = 1, h_2 = \rho, h_3 = \rho \sin \phi\).

See (37) on page 1077 and (70) on page 1081 of Greenberg for the explicit formula for cylindrical polar and spherical Laplacians.

### 2.2 Separation of Variables

**§20.1:** When the Laplacian of \(u\) equals to zero, the equation for \(u\) is called the Laplacian equation. When the Laplacian of \(u\) equals to a “source function” (a known function of position), then the equation for \(u\) is called the Poisson’s equation. Generally speaking, Laplacian problems deal with a volume domain of interest. Boundary condition must be
specified on the surface of the volume—including the “surface” at infinity if the volume domain extends to infinity. When the value of $u$ over all of the surface of the volume of interest is freely specified, the problem is said to be a Dirichlet Problem. When the “normal component” of the gradient of $u$ on the surface of the volume of interest is specified, the problem is said to be a Neumann Problem. It is of interest to note that the Neumann boundary condition must satisfy a certain compatibility condition (ask me about it if I forgot to mention it in class). It is also possible for the boundary condition to be “mixed.”

Take a look at Problem 18(a) on page 1069. The uniqueness proof for the Laplace equation is very, very similar to the one I gave for the diffusion problem.

It is of interest to note that the idea of “marching” away from an “initial condition line” which served us well in the Diffusion and in the Wave problems does not work here.

§20.2: Nothing significantly new arises as far as the separation of variables is concerned. Look at (4a) and (4b) on page 1059: when you use Cartesian coordinates, one function is trigonometric, and the other is exponential. You have to pick which one is which.

Study Fig. 4 on page 1063—the trick is to decompose the original problem into the sum of a number of simpler problems—essentially the same trick outlined in Notes 7c for the wave equation. Remember the word SUPERPOSITION.

Non-Cartesian Coordinates: When you separate the variables in cylindrical polar coordinates, you get (5) and (6) on page 1071. Luckily, the $R(r)$ function is of the “Cauchy-Euler” form, and its exact solution can be written down (see (7) on page 1071). Take care to note how the separation constant $\kappa$ is determined (just like all the other separation of variables problems!).

Scan §20.3.2, where you meet Bessel function again. Go on to read §20.3.3. What happens here? You meet $\Phi(\phi)$ which obeys the Legen-

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1When some (or all) of the boundary is at “infinity,” the condition at the infinity surface may be something qualitative such as “boundedness.” Essentially, the test is: the solution found should be the “limiting” solution as the infinity boundary moves further and further out.
A equation (77) on page 1082. Its **bounded** solutions are called Legendre polynomials (see §4.4 of Greenberg; in particular the paragraph above (6) on page 213 justifying (6), and Table 1 on the same page).

**§20.4:** Fourier Transform. We will skip this section—it is quite straightforward.

**§20.5:** Numerical Solution. (5) on page 1093 is the bottom line! We will skip §20.5.2. Read §20.5.3, to appreciate once again the power of iterations!. (25) on page 1100 is called the Jacobi scheme; (26) on page 1101 is called the Gauss-Seidel scheme; and (28) on page 1101 is called the SOR scheme, with ω a parameter to mess around with.

### 2.3 Problems

1a , page 1067. Straightforward.

10a , page 1068. Straightforward

5 , page 1085. What happens if you remove the innocent looking requirement “u bounded as $r \to \infty$”? What is the solution for the special case $f(\theta) = 1$; in particular, what is the value of $u$ at infinity?

**Lam Special:** Do essentially the same problem immediately above, except for the three dimensional case. Using spherical coordinates $\rho, \phi, \theta$, the volume of interest is the domain outside the sphere $\rho = a$. The boundary condition on this sphere is $u(\rho = a, \theta, \phi) = 1$. The boundary condition at infinity is $u(\rho \to \infty, \theta, \phi) = 0$. What is the solution? Note that the two-dimensional problem above does NOT permit this kind of “infinity” boundary condition but it is acceptable for three dimensional problems.

1j , page 1103. Take advantage of symmetry, so that you only need to deal with 4 unknowns. Just write down the algebraic equations—you don’t need to solve them if you don’t feel like it. If you know Matlab and feel like it, solve it. What is the effect on the numerical solution if the hole is not round, but it does pass through the four grid points?