MAE 306 Notes #8b
Engineering Mathematics II

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1 Supplementary Notes

There are three well-known linear elliptic PDEs associated with the Laplace operator ∇^2, defined as $\nabla \cdot \nabla$ (in one, two or three dimensions):

$$\nabla^2 u = 0, \qquad \text{Laplace Equation} \tag{1}$$

$$\nabla^2 u = -\lambda^2 u, \qquad \text{Helmholtz Equation} \tag{2}$$

$$\nabla^2 u = f(\mathbf{x}), \qquad \text{Poisson Equation.} \tag{3}$$

You are reminded once more that separation of variables is applicable only if the domain of interest is "separable."[1] What do you do if your problem does not allow the use of separation of variables (besides running toward the nearest computer)?

2 Eigen Function Approach

Consider a general Poisson Equation Dirichlet problem for u in some "volume" domain of interest. Let us define $u_1(\mathbf{x})$ by:

$$u = F(\mathbf{x}) + U(\mathbf{x}, t) \tag{4}$$

where $F(\mathbf{x})$ is some twice differentiable scalar function that takes on the desired boundary values for u on the boundary surface—and is otherwise

[1] See Morse and Feshbach, Method of Mathematical Physics, Vol. 1, McGraw Hill, pp. 495-522.

arbitrary. Now, the PDE to be satisfied by U remains a Poisson Equation:

$$\nabla^2 U = -h(\mathbf{x}) \tag{5}$$

where $h(\mathbf{x}) = \nabla^2 F - h(\mathbf{x})$ is the "modified" inhomogeneous (source) term and is a known function of $\mathbf{x}$. The boundary condition for U on the boundary surface is zero. Once more, you see how we remove non-zero boundary conditions at the cost of modifying the inhomogeneous term.

Let us rewrite this equation as follows:

$$\mathcal{L}(U) = h(\mathbf{x}) \tag{6}$$

where

$$\mathcal{L}(.) \equiv -\nabla^2(.) \tag{7}$$

is a self-adjoint linear (second order differential) operator.[2]

Before proceeding further, we pause to find the eigenfunctions of the $\mathcal{L}$ operator $\phi_n(\mathbf{x})$ and eigenvalues λ_n defined by:

$$\mathcal{L}(\phi_n) = \lambda_n \phi_n. \tag{8}$$

The Helmholtz equation naturally comes in to introduce itself here! The eigen functions are ordered by the subscript n in ascending magnitude of the eigenvalues.

Following the general ideas used in the Sturm-Liouville Theory, we can prove that $\mathcal{L}$ is a positive-definite operator, that there is a lower bound for the magnitude of λ_n, and that $\lambda_n \to \infty$ for $n \to \infty$. Most importantly, the eigenfunctions are orthogonal to each other:

$$< \phi_m, \phi_n >= \delta_{m,n} \tag{9}$$

where $\delta_{m,n}$ is called the Kronecker Delta: it is zero when $m \neq n$, and is unity (by appropriate normalization) when $m = n$. The "inner product" now involves integration over the whole spatial volume of interest.

The forcing (source term) can now be expanded in terms of the eigenfunctions (serving as our basis functions):

$$h(\mathbf{x}) = \sum_{n=1}^{\infty} A_n \phi_n(\mathbf{x}). \tag{10}$$

[2]To find the adjoint operator and to show that it is self-adjoint (in more than one dimension), see §5.

The coefficients A_n's can readily be computed from:

$$A_n =< \phi_n(\mathbf{x}), h(\mathbf{x}) > . \tag{11}$$

Hence, the Poisson Equation for U is now:

$$\mathcal{L}(U) = \sum_{n=1}^{\infty} A_n \phi_n(\mathbf{x}). \tag{12}$$

In order to solve for U, we first expand U in terms of our basis functions:

$$U = \sum_{n=1}^{\infty} U_n \phi_n(\mathbf{x}) \tag{13}$$

where the U_n's are unknown constants. Substituting it in (12), we obtain:

$$\mathcal{L}(\sum_{n=1}^{\infty} U_n \phi_n(\mathbf{x})) = \sum_{n=1}^{\infty} A_n \phi_n(\mathbf{x}), \tag{14}$$

or, since $\mathcal{L}(\phi_n) = \lambda_n \phi_n$, we obtain:

$$\sum_{n=1}^{\infty} U_n \lambda_n \phi_n(\mathbf{x}) = \sum_{n=1}^{\infty} A_n \phi_n(\mathbf{x}). \tag{15}$$

This is one single algebraic equation for infinite number of unknowns, the U_n's—and it must hold for any value of $\mathbf{x}$. Taking the inner product with $\phi_m(\mathbf{x})$, we obtain:

$$U_m = \frac{A_m}{\lambda_m}, \quad m = 1, \ldots, \infty. \tag{16}$$

Voila! The problem is solved.

In the above abstract presentation, it is assumed that somehow the eigenfunctions and eigenvalues can be found, perhaps one by one numerically, if all else fails. The methodology of separation of variable is not mentioned in the exposition at all, but if the geometry of the volume of interest should allow it, then separation of variables should definitely be helpful in finding the eigenfunctions and eigenvalues. Otherwise, we will have to do it some other way.

3 Green's Function Approach

Consider the Green's function G for the specific volume domain of interest. The governing PDE is:

$$\mathcal{L}(G) = \delta(\mathbf{x} - \mathbf{x}_o) \tag{17}$$

where $\delta(\mathbf{x} - \mathbf{x}_o)$ is our old friend, the Dirac Delta function (it is zero everywhere except in the neighborhood of its argument near zero; the volume integral over this little speck of volume of the Dirac Delta function is one.) The point $\mathbf{x}_o$ is, as discussed several lectures ago, the "obervation point." The Green's function G is thus a function of both $\mathbf{x}$ and $\mathbf{x}_o$: $G = G(\mathbf{x}; \mathbf{x}_o)$. The "boundary" of the volume domain of interest is a surface. If we have a Dirichlet Problem for U, then we require that G be zero there; if we have a Dirichlet Problem for U, then we require $\mathbf{n} \cdot \nabla G$ be zero there.

Taking the inner product of the Poisson's equation of interest for U with G_∞, and subtracting from it the inner product of U with (17), we have (this is the standard trick!):

$$< \mathcal{L}(U), G_\infty > - < U, \mathcal{L}(G_\infty) > = < h(\mathbf{x}), G_\infty > - < U, \delta(\mathbf{x} - \mathbf{x}_o) > \tag{18}$$

The left-hand side can be expressed in terms of a surface integral—for details, see §5. The second term on the right hand side is a god-send—remember, a Dirac Delta function is a wonderful gift! Voila! the solution is found:

$$U(\mathbf{x}_o) = < h(\mathbf{x}), G(\mathbf{x}; \mathbf{x}_o) > + \iint_{\mathcal{A}} \mathbf{n} \cdot [G \nabla U - U \nabla G] \, d\mathcal{A}. \tag{19}$$

The $\mathbf{x}$ on the inner product term on the right-hand side serves as the dummy variable in the integration involved in the inner product. Physically, the first term represents the effects of $h(\mathbf{x})$, while the second term, which involves a surface integral, represents the effects of boundary conditions.

The surface integral appears to require knowledge of both U and $\mathbf{n} \cdot \nabla U$ on the boundary surface. But this is not so! For Dirichlet Problems, the value of ∇U on the boundary surface is unknown, but $G = 0$ there so $G \nabla U$ vanishes there. For Neumann Problems, the value of U on the boundary surface is unknown, but $\nabla G = 0$ there so $U \nabla G$ vanishes there. In short, everything that is needed on the right hand side is known—so, in the parlance of mathematicians, the solution is reduced to "quadrature"—provided the relevant Green's function is available.

3.1 Explicit Fundamental Green's Functions

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In three dimension (x, y, z), the infinite domain "fundamental Green's Function" G_∞ is:

$$G_\infty(\mathbf{x};\mathbf{x}_o) = \frac{1}{4\pi}\frac{1}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}} \tag{20}$$

For finite $\mathbf{x}_o$, we see that G_∞ vanishes as $\mathbf{x} \to \infty$.

In two dimension (x, y), G_∞ is:

$$G_\infty(\mathbf{x};\mathbf{x}_o) = \frac{1}{2\pi}\log\sqrt{(x-x_o)^2+(y-y_o)^2} \tag{21}$$

For finite $\mathbf{x}_o$, we see that G_∞ does not vanish as $\mathbf{x} \to \infty$, but ∇G_∞ does. So our "bilinear concommitant" term is actually OK. In any case, we see that we need to be extra careful for two-dimensional problems.

In one dimension, G_∞ does not exist. If the domain of interest is finite, however, the Green's function is readily obtained.

The method of images can be useful in the Green's function approach when the volume of interest is not the full spatial volume. For example, if the volume of interest is the three dimensional half space defined by $x \geq 0$ and we have a Dirichlet Problem, the relevant Green's function is:

$$G = G_\infty(x, y, z; x_o, y_o, z_o) - G_\infty(x, y, z; -x_o, y_o, z_o). \tag{22}$$

If for the same half space domain we have a Neumann Problem, the relevant Green's function is:

$$G = G_\infty(x, y, z; x_o, y_o, z_o) + G_\infty(x, y, z; -x_o, y_o, z_o). \tag{23}$$

4 The Method of Singularities

Then there is the tinker toy approach, used extensively in aerodynamics, and is sketched briefly in Problem 7 on page 1085. I will talk a little on this in class.

5 Self-Adjointness of L

Consider:

$$< \mathcal{L}(u), v >= \iiint_{\mathcal{V}} -(\nabla \cdot \nabla u) v d\mathcal{V} \tag{24}$$

It can be rewritten as, after some careful vector calculus:

$$\begin{aligned} < \mathcal{L}(u), v > &= \iiint_{\mathcal{V}} -\nabla \cdot (v \nabla u) d\mathcal{V} + \iiint_{\mathcal{V}} (\nabla u) \cdot (\nabla v) d\mathcal{V} & (25) \\ &= \iiint_{\mathcal{V}} \nabla \cdot (u \nabla v - v \nabla u) d\mathcal{V} + < u, \mathcal{L}(v) > . & (26) \end{aligned}$$

The operator $\mathcal{L}$ is thus shown to be self-adjoint.

Using the Divergence Theorem on the first term, we have:

$$< \mathcal{L}(u), v > - < u, \mathcal{L}(v) >= \iint_{\mathcal{A}} \mathbf{n} \cdot [u \nabla v - v \nabla u] \, d\mathcal{A} \tag{27}$$

where $\mathbf{n}$ is our old friend, the unit outward normal of the surface element $d\mathcal{A}$. Hence, if both u and v vanish on the boundary surface, or if their gradients vanish on the boundary surface (which may be at infinity), then we have

$$< \mathcal{L}(u), v >=< u, \mathcal{L}(v) > \tag{28}$$

6 Matlab PDE Toolbox

You can go to

http://www.mathworks.com/products/pde/

to take a look at Matlab's PDE Toolbox (for two spatial dimensions only, plus time). The $199.00 software capability includes:

$$\begin{aligned} &\text{Elliptic Problems} & -\nabla \cdot (c \nabla u) + au = f, && (29) \\ &\text{Parabolic Problems} & d\frac{\partial u}{\partial t} - \nabla \cdot (c \nabla u) + au = f, && (30) \\ &\text{Hyperbolic Problems} & d\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c \nabla u) + au = f, && (31) \\ &\text{Eigenvalue Problems} & -\nabla \cdot (c \nabla u) + au = d\lambda u, && (32) \end{aligned}$$

where c, a, d, and f may be complex valued scalar functions of x and y; for the parabolic and hyperbolic case, they may even depend on t. A non-linear solver option is provided to handle weak u dependence. Solving problems within this repertoire amounts to clicking some boxes, defining the geometry of the domain of interest, type in the initial and boundary conditions, and using a provided mesh generator to generate the mesh. And then you click something, and you are ready to see the computed answers using the plotting routines.

Intellectually, the “difficult” part of this software package is the handling of the irregular “mesh.” How to numerically compute (an approximation to) gradient and divergence when you have a mesh of finite discretness and irregular shapes is something that is not immediately obvious. If you are interested, ask me.

I do not own the PDE Toolbox (I have a Mac, and Matlab has stopped supporting Macs). Perhaps it is available on some of the University non-Mac computers.