

MAE 306 Notes #9a

Engineering Mathematics II

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1 Reading and Homework Assignments

Study Chapter 21, Complex Variable, pp. 1108-1149 of Michael Greenberg's *Advanced Engineering Mathematics*, and do the problem assigned below.

The problems are due on Tuesday, April 13th, 1999, 5PM. Please submit your homework to the MAE 306 homework **IN** tray outside D-302, E.Q.

1.1 Comments on Readings and Problems

§21.1, 21.2: Pretty straightforward stuff. See (13) on page 1111 for definition of \bar{z} , the complex conjugate of z .

- Problem 6b on page 1113. Making sure you know what complex conjugate is.
- Problem 9c on page 1113. Greenberg wants you to find the real part and the imaginary part.

§21.3: Note the definition of *open* and *closed regions*. Note also the fancy word *mapping*. On page 1117, you finally see a “proof” of (7)—the Euler Formula—and (8). On page 1118, you see the trigonometric and hyperbolic functions defined side-by-side. You should know (20a,b,c,d) by heart. §21.3.4 is a minor application of what we have learned so far. It is not profound.

- Problem 9a on page 1124,

- Problem 9d on page 1124,
- Problem 10 on page 1124; make sure you remember (20a,b,c,d) on page 1118.

§21.4: Equation (2) on page 1126 is the big deal. You may agree with me that Mr. de Moivre's (9) on page 1127 is a cheap way to become immortalized. On page 1128, there is an obvious typo on the second line of the bottom paragraph: one of the exponents is wrong. Equation (17) on page 1129 is important; note that $k = 0, \pm 1, \pm 2, \dots$ tag along. §21.4.6 introduces branch cuts—a way to render a multi-valued function single-valued by agreeing not to cross certain lines drawn on the sand.

- Problem 5a, 5b and 5d on page 1134.
- Problem 6a, 6c on page 1134.

§21.5. Equation (8) on page 1138 is a big deal. Look at Example 4! \bar{z} is NOT an analytic function! Equation (18) on page 1140 is a very big deal. Note in the middle of page 1142: if a function is not analytic at z_o , then it is said to be *singular* at z_o . Note the polar coordinate form of the Cauchy-Riemann condition (30) on page 1144. Look at the spectacular Theorem 21.5.2 on page 1145.

- Problem 5a, 5d on page 1147. To assure you that finding $f'(z)$ is no big deal.
- Problem 11e on page 1147, (how many and where are the singular points?)
- Are the $f(z)$'s given in Problems 11b and 11f analytic functions?
- Problem 16a only. Hint: the normal vector to $u(x, y) = \text{constant}$ is parallel to ∇u ; the normal vector to $v(x, y) = \text{constant}$ is parallel to ∇v . If these two normal vectors are mutually perpendicular to each other, what do you get if you take their dot product?¹

While the number of problems assign is relatively large, most of them are very short problems.

¹The “mesh” formed by $u(x, y) = \text{constant}$ and $v(x, y) = \text{constant}$ are wonderful as the mesh for numerical solution of PDEs! They are “curvilinear coordinates” which are “locally Cartesian.”

2 Supplementary Notes on Complex Variable

The square root of minus one is denoted by i , affectionately called the *imaginary number* by all. Hence, by definition, $i^2 = -1$. With a bit of work, you can prove that $\sqrt{i} = \pm(1+i)/\sqrt{2}$.

Consider now an arbitrary (real or complex) function F which depends on two independent Cartesian variables x and y : $F = F(x, y)$. We now introduce a new set of independent (complex) variables z and \bar{z} by:

$$z = x + iy, \quad (1a)$$

$$\bar{z} = x - iy. \quad (1b)$$

\bar{z} is commonly referred to as the *complex conjugate* (or simply “conjugate”) of z . The inverse of this transformation is:

$$x = \frac{z + \bar{z}}{2}, \quad (2a)$$

$$y = \frac{z - \bar{z}}{2i}. \quad (2b)$$

In general, then, any function $F(x, y)$ can be transformed into $F(z, \bar{z})$.

The subject of the theory of complex variable is focused on the class of F which, when expressed in terms of z and \bar{z} , is independent of \bar{z} . In other words, the focus is on the restricted class of F which depends solely on z —*i.e.* $F = F(z)$.

In general, F is complex (when expressed in terms of x and y), and can be expressed in the following form:

$$F(z) = U(x, y) + iV(x, y) \quad (3)$$

where $U(x, y)$ and $V(x, y)$ are real functions of the real independent variables, x and y . It is trivially easy to dream up such functions: just dream up any function $F(x)$, and you replace x by z . And voila! you have just invented a $F(z)$, the kind that we shall study for the rest of this semester!

- The function $F(z)$ is said to be *analytic* in a domain of interest if
 1. it exists,
 2. it is single-valued, and
 3. its derivative with respect to z , denoted by $F'(z)$, exists.

- An “interesting” property of an analytic function is that $F'(z)$, the derivative of $F(z)$ with respect to z , defined by:

$$\frac{dF}{dz} = \lim_{\Delta x, \Delta y \rightarrow 0} \frac{F(z_2) - F(z_1)}{(x_2 - x_1) + i(y_2 - y_1)}, \quad (4)$$

is independent of the “direction” along which the limiting process is taken. For example, the following yields the identical value for $F'(z)$.

$$F'(z) = \frac{dF}{dz} \quad (5a)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{F(x_2 + iy_1) - F(x_1 + iy_1)}{(x_2 - x_1)}, \quad (5b)$$

$$= \lim_{\Delta y \rightarrow 0} \frac{F(x_1 + iy_2) - F(x_1 + iy_1)}{i(y_2 - y_1)}, \quad (5c)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{F((x_1 + \Delta x) + i(y_2 + \gamma \Delta x)) - F(x_1 + iy_1)}{\Delta s + i\gamma \Delta x}, \quad (5d)$$

where γ in (5d) is any arbitrary number. If $F'(z)$ satisfies the requirements of an analytic function, then it is an analytic function.

- A spectacular consequence of the above interesting property is:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad (6)$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \quad (7)$$

These two relations are called *Cauchy-Riemann Relations*.

- An incredible consequence of the above spectacular result is:

both $U(x, y)$ and $V(x, y)$ are exact solutions to the two-dimensional Cartesian Laplace Equations.

This serves as the foundation for the methodology of *Conformal Mapping* for finding solutions to two-dimensional Laplace Equations.

- The gradient of F is given by:

$$\nabla F = \mathbf{e}_x \frac{\partial F}{\partial x} + \mathbf{e}_y \frac{\partial F}{\partial y} \quad (8)$$

and the notations used are standard and obvious. The differential dF , defined as the difference of the values of F at x_1, y_1 and at x_2, y_2 which are infinitesimally apart, is given by:

$$dF = \nabla F \cdot d\mathbf{s} \quad (9)$$

where $d\mathbf{s}$ is the infinitesimal displacement vector

$$d\mathbf{s} = \mathbf{e}_x dx + \mathbf{e}_y dy; \quad dx = x_2 - x_1, \quad dy = y_2 - y_1. \quad (10)$$

When $F = F(z)$, the gradient vector of $F(z)$ is, by the chain rule:

$$\nabla F = F'(z)(\mathbf{e}_x + i\mathbf{e}_y) \quad (11)$$

Hence, using ∇F from (11) and $d\mathbf{s}$ from (10), dF can be written as follows:

$$dF = \nabla F \cdot d\mathbf{s} = F'(z)dz \quad (12)$$

If we integrate from z_i to z_f along any path of our choice in the x, y plane, we have:

$$\int_{x_i+iy_i}^{x_f+iy_f} \nabla F \cdot d\mathbf{s} = \int_{z_i}^{z_f} F'(z)dz = F(z_f) - F(z_i). \quad (13)$$

The important point to note is that the result of the line integral is independent of the integration path chosen (provided the paths chosen can be “deformed” from one to the other); the answer depends only on the coordinates of the initial and the final points. For fixed terminal points, the value of the line integral is unchanged if the integration path is changed. What happens if the integration path chosen is a closed loop—the starting point and the final points are the same points—and the integrand is totally analytic in the closed-region defined by the loop? The value of the line integral is then ZERO! (provided you can “deform” the path into a nothing path).

- Consider the closed-loop integral:

$$A = \oint G(z)dz \quad (14)$$

where $G(z)$ is analytic on and inside the domain bounded by the closed-loop path of integration. Since any analytic $G(z)$ is actually the derivative of some analytic function (such as $F'(z)$), this line integral A must

be identically zero. This result is called the *Cauchy Theorem*. Personally, I think this short “proof” is prettier than the one presented in Greenberg in §23.3 on page 1189 later.

- How about

$$B = \oint \frac{G(z)}{z - a} dz \quad (15)$$

where a is a complex parameter inside the closed-loop path which is part of the definition of B ? Since the value of B is unchanged when the integration path is “deformed,” we simply “shrink” it so that the path wraps around the point $z = a$. Make that path a circle with radius ϵ about the point. For very, very small ϵ , the value of $G(z)$ on the integration path is a constant, thus we have:

$$B = G(a) \lim_{\epsilon \rightarrow 0} \oint \frac{dz}{z - a}. \quad (16)$$

The integration can easily be carried out, yielding:

$$B = \frac{G(a)}{2\pi i}. \quad (17)$$

This result is called the *Residue Theorem*.

- The above exposition gives you an overview of the whole foundation of the beautiful theory of complex variables.

The major buzz words for the rest of the semesters are:

1. i , the imaginary number,
2. Complex conjugate,
3. Analytic function,
4. Cauchy-Riemann Relations,
5. Conformal Mapping,
6. Cauchy Theorem,
7. Residue Theorem.

You are expected to know by heart what they are.