1. The substantial derivative of a scalar such as \( \rho \) is denoted by \( \frac{D \rho}{D t} \). In Cartesian coordinates, we have:

\[
\frac{D \rho}{D t} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}
\]

\((u,v,w\ are\ the\ components\ of\ \mathbf{V}\ in\ the\ e_x,\ e_y,\ e_z\ directions,\ respectively)\)

In cylindrical polar coordinates, we have:

\[
\frac{D \rho}{D t} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \frac{v}{r} \frac{\partial \rho}{\partial \theta} + w \frac{\partial \rho}{\partial z}
\]

\((u,v,w\ are\ now\ the\ components\ of\ \mathbf{V}\ in\ the\ e_r,\ e_\theta,\ e_z\ directions,\ respectively)\).

Now, we are interested in \( \frac{D \mathbf{V}}{D t} \).

(1a) Write down the three Cartesian components of \( \frac{D \mathbf{V}}{D t} \).

(1b) Write down the three cylindrical polar components of \( \frac{D \mathbf{V}}{D t} \). You must first work out (by yourself) what the partial derivatives of \( e_r, e_\theta, e_z \) with respect to \( r, \theta \) and \( z \) are.

(1c) Write the inviscid momentum equation

\[
\rho \frac{D \mathbf{V}}{D t} = - \nabla p
\]

in cylindrical polar coordinates.

2. The continuity, momentum and energy conservation equations for an inviscid fluid are:

\[
\frac{D \rho}{D t} + \rho \nabla \cdot \mathbf{V} = 0,
\]

\[
\frac{D \mathbf{V}}{D t} = - \frac{1}{\rho} \nabla p,
\]

\[
\frac{D \left( h + \frac{V^2}{2} \right)}{D t} = \frac{1}{\rho} \frac{\partial p}{\partial t}.
\]

Justify the following statement: the steady flow solution of this set of equations is independent of scale (or size) of the flow field.
3. Read the derivation of the speed of sound in Shapiro on page 45-47. Note that in the derivation, only the momentum and the continuity equation were used explicitly. On page 47, Shapiro recommends that $dp/dp$ be replaced by $(\partial p/\partial \rho)_s$ by mumbling some justification.

   (a) Can you provide a more satisfactory justification than he gave (e.g. mention Reynolds number)?

   (b) If you blindly make the assumption that the flow is isothermal, how would you go about finding the resulting speed of sound?

4. You are given a very large tank containing compressed air (assumed to be a perfect gas with ratio of specific heats=1.4) at 1.2 atmosphere and 20 °C. It is "connected" to the outside by a smooth (one-dimensional) DeLaval nozzle with throat area $A_t$ of reasonable size and exit area $A_e$ ($A_e > A_t$ assumed).

   (a) What happens to the mass flow rate $\dot{m}$ when $A_e$ increases while $A_t$ is held fixed? When happens to the mass flow rate $\dot{m}$ when $A_e$ is held fixed while $A_t$ is increased? Plot or sketch $\dot{m}$ as a function of $A_e/A_t$, and use it to help you discuss this problem. There is a critical value of $A_e/A_t$ across which the response of the mass flow to variation of $A_e/A_t$ changes character. Find this value. What physically happens to the flow in the nozzle at this value?

   (b) What happens to the mass flow rate if the tank pressure remains at 1.2 atmosphere, but the temperature of the compressed air is now 600 °C instead of 20 °C?

   (c) The $A_e/A_t$ of the given nozzle is 2.70. Find the pressure in the tank so that the flow is choked at the throat. Does this answer depend on the temperature in the tank? (the pressure outside the tank is 1 atm).

   (d) It turns out that the tank is not very, very, very large, so that the pressure in the tank is not constant but a function of time. The diameter of the throat is 1cm, and the pressure in the tank drops by 10% in one second. What is your estimate of the error of the steady state assumption?

   (e) When the tank pressure drops by 10%, what is your estimate of the percentage drop of the tank temperature?