Look into the reference books to find out what Biot-Savart Law is. Just know what it is. Not the derivation.

Let the 2-dimensional flow field be described by the some of an irrotational flow \( \phi_o(x,y) \) which is smooth in the neighborhood of the origin plus a potential vortex of strength \( \Gamma \) located at the origin. We have

\[
\phi(x,y) = \phi_o(x,y) + \frac{\Gamma}{2\pi} \theta .
\]  

(1)

The velocity field is given by:

\[
V(x,y) = U_o(x,y) + \frac{\Gamma}{2\pi r} e_\theta .
\]  

(2)

where \( U_o(x,y) \equiv \nabla \phi_o \) and \( e_\theta \) is the unit vector in the \( \theta \) direction.

Now let us compute the external force per unit span (\( F \)) which is generated by this vortex. We have for any control volume that encloses the vortex:

\[
-\mathbf{F} + \text{resultant of pressure forces}=\text{net outflux of momentum}.
\]  

(3)

Note: there is a minus sign in front of \( F \) because \( -F \) is the force required to hold the control volume in place. Let us consider the control volume to be a cylinder with its center at the origin and its radius is \( \varepsilon \):

- Net outflux of momentum = \[
\oint (\mathbf{n} \cdot \mathbf{V}) \mathcal{V} dS = \int_0^{2\pi} (\mathbf{e}_r \cdot \mathbf{V}) \mathcal{V} e\theta d\theta
\]  

(3)

where \( e_r \) is the unit vector in the radial direction. It is easy to show that this term tends to zero as \( \varepsilon \) tends to zero.

- Resultant of pressure forces acting on the surface of control volume =

\[
\int n p dS = \int_0^{2\pi} e_r p e\theta d\theta .
\]  

(4)
Now we get the pressure $p$ from the incompressible Bernoulli's equation:

$$p = p^o - \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v}$$  \hspace{1cm} (5)$$

where $p^o$ is the Bernoulli constant (independent of $x,y$). Using (2), we obtain:

$$p = \left( p^o - \frac{\rho}{2} \mathbf{u}_o \cdot \mathbf{u}_o \right) + \left( \frac{\Gamma^2}{4\pi^2 r^2} + \frac{\Gamma \mathbf{u}_o \cdot \mathbf{e}_r}{\pi r} \right).$$  \hspace{1cm} (6)$$

The first term is not a constant on the surface of this control volume, but its contribution tends to zero as $\varepsilon$ tends to zero. The second term is large when $\varepsilon$ is small, but it is a constant on the surface of this control volume (aren't we smart to pick this!), thus it can be explicitly shown to be zero. The third term has a finite limit when $\varepsilon$ is asymptotically small. And a detailed calculation yields the result that

$$F = \text{lift per unit span, directed according to right hand rule as explained in class } = \rho \mathbf{U}_o(0,0) \Gamma.$$  

**Problem set #8**

10/22/1995

1. Work out the detailed calculations of the third term in the above Joukowski lift law derivation.

2. Show that there is zero drag acting on this vortex.

3. Consider a two or three dimensional flow field consisting of some irrotational flow $\phi_o(x,y,z)$ plus a source of strength $Q$ located at the origin. Find the force generated by the source, showing that it is a thrust (acting in the direction opposite to the velocity vector $\mathbf{U}_o(0,0,0)$ with amplitude

$$F = \rho \mathbf{Q} \mathbf{U}_o(0,0,0).$$

**Hint:** for 3-dimensional flow, you choose a sphere for your control volume. Note the elemental (ring) area $dS$ of a sphere is $dS = 2\pi r^2 \sin \theta \ d\theta$ where $\theta$ is the angle from the pole. The algebra is somewhat messy in trying to prove certain integrals to be zeros.