Here is an integral equation for $\gamma(x)$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\gamma(\xi)d\xi}{\xi - x} = B \quad (I)$$

where $B$ is a constant. We need to say that the principle value of the improper integral is to be used on the right hand side (my word processor doesn’t let me do that).

Let's change the variables as follows:

$$x = \cos \theta, \quad \xi = \cos \phi,$$

$$g(\phi) = \gamma(\xi) \sin \phi = \gamma(\xi) \sqrt{1 - \xi^2}.$$

We now have an integral equation for $g(\phi)$:

$$\frac{1}{\pi} \int_{0}^{\pi} \frac{g(\phi)d\phi}{\cos \phi - \cos \phi_o} = B \quad (II)$$

The following is usually called the Glauert Identity (this can be proved using contour integration):

$$\int_{0}^{\pi} \frac{\cos(n\phi)d\phi}{\cos \phi - \cos \phi_o} = \frac{\pi \sin(n\phi_o)}{\sin \phi_o}, \quad n = \text{integers (including zero)}.$$

So, the solution for $g$ is:

$$g(\phi) = A + B \cos \phi$$

where $A$ is an arbitrary constant, and is the homogeneous solution of (II). The $B \cos \phi$ term is the inhomogeneous solution. In other words, the complete solution for $\gamma(x)$ is:

$$\gamma(x) = \frac{A}{\sin \phi_o} + \frac{B \cos \phi_o}{\sin \phi_o} \sqrt{1 - x^2}.$$
Note that $\gamma(x)$ is singular either at $x=-1$ or $x=+1$. We can suppress one of the two singularities by choosing $A$ appropriately.

If we want to suppress the singularity at $x=1$, we choose $A=-B$. Thus we have:

$$\gamma(x) = -B\sqrt{\frac{1-x}{1+x}}$$

Integrating this with respect to $x$ from $-1$ to $+1$, we obtain:

$$\int_{-1}^{1} \gamma(x)dx = -B\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}dx = -B\pi$$

If $B$ is a function of $x$, or, in terms of $\theta_0$, is a function of $\theta_0$, then the solution for $g(\theta)$ can similarly be constructed: $g(\theta)$ can be represented by a cosine Fourier series, and the Glauert identify can be used to determine the coefficients in terms of $B(\theta)$.

Homework Problem #9

Consider a thin airfoil of chord $c$ located with its mid-chord point at the origin. We are not concerned with the thickness problem at all here. The camber line is defined by:

$$y = 0, \quad -c/2 \leq x \leq +c/4,$$

$$y = -\alpha, \quad +c/4 \leq x \leq +c/2.$$ 

In other words, this is a flat plate airfoil with a flap starting at the 3/4 chord point. The flow is from left to right.

Find the lift of this problem using the minimum number of terms of the Fourier series. (Hint: there is something subtle or cute here).