

Notes #10

MAE 533, Fluid Mechanics

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1 Vorticity

The vector field $\boldsymbol{\Omega}$ defined by:

$$\boldsymbol{\Omega} \equiv \nabla \times \mathbf{q} \quad (1)$$

is called *vorticity*. It has the physical meaning of being twice the *averaged* angular velocity of a small element of fluid. In Cartesian coordinates, the formula to compute $\boldsymbol{\Omega}$ is:

$$\boldsymbol{\Omega} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \quad (2)$$

where \mathbf{e}_x , \mathbf{e}_y , \mathbf{e}_z and u , v , w are unit vectors and velocity components in the x , y , z directions, respectively. In cylindrical polar coordinates, the formula is:

$$\boldsymbol{\Omega} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u & rv & w \end{vmatrix} \quad (3)$$

where u , v , w are the velocity components in the r , θ and z directions, respectively. In the above definition, the velocity vector field $\mathbf{q}(\mathbf{x})$ is assumed differentiable. If you have a discontinuous velocity field, you can imagine it to be represented by a continuous function (rapidly varying at the discontinuity), and compute the vorticity accordingly.

2 The Vorticity Equation

The inviscid momentum equation is:

$$\frac{D\mathbf{q}}{Dt} = -\frac{1}{\rho}\nabla p. \quad (4)$$

Using the vector identity:

$$\mathbf{q} \cdot \nabla \mathbf{q} = \nabla \left(\frac{\mathbf{q} \cdot \mathbf{q}}{2} \right) - \mathbf{q} \times \boldsymbol{\Omega}, \quad (5)$$

we can rewrite (4) as:

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \boldsymbol{\Omega} = - \left(\frac{\nabla p}{\rho} + \nabla \frac{\mathbf{q} \cdot \mathbf{q}}{2} \right). \quad (6)$$

We now take the curl of (6) to obtain:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \mathbf{q} \cdot \nabla \boldsymbol{\Omega} + \boldsymbol{\Omega} (\nabla \cdot \mathbf{q}) - \boldsymbol{\Omega} \cdot \nabla \mathbf{q} = -\nabla \left(\frac{1}{\rho} \right) \times \nabla p. \quad (7)$$

Using the continuity equation, we can simplify (7) to yield:

$$\frac{D}{Dt} \left(\frac{\boldsymbol{\Omega}}{\rho} \right) = \frac{\boldsymbol{\Omega}}{\rho} \cdot \nabla \mathbf{q} + \frac{1}{\rho^3} \nabla \rho \times \nabla p. \quad (8)$$

Equation (8) is known as the **vorticity equation**.

2.1 Barotropic Flows

The second term on the right hand side of (8) is called the *baroclinic term*. It is non-zero whenever $\nabla \rho$ and ∇p are both non-zero *and* are not parallel with each other.

A flow field in which density can be shown to be solely a function of pressure is called a *barotropic flow*. It is trivial to show that for barotropic flows, the baroclinic term is zero (the two vectors in question are parallel).

When can we say we have a barotropic flow field? The most common case is when we have an *iso-compositional* flow field (the same equation of state prevails everywhere), *and* at the same time the entropy s is the same everywhere. A more contrived example is an iso-compositional flow field where

the temperature is known to be a constant (because the heat conductivity is very large)

For barotropic flows, the vorticity equation is:

$$\frac{D}{Dt} \left(\frac{\boldsymbol{\Omega}}{\rho} \right) = \frac{\boldsymbol{\Omega}}{\rho} \cdot \nabla \mathbf{q}. \quad (9)$$

The term on the right hand side is always zero in two-dimensional flow. It represents the effects of stretching and bending of vortex lines.

2.2 Conditions for Irrotational Flows

By inspection, $\boldsymbol{\Omega} = 0$ is an exact solution of (9). Flows with $\boldsymbol{\Omega} = 0$ are called *irrotational flows*, or *potential flows*.

When can we conclude that a given fluid flow problem is irrotational? Here is the answer:

An inviscid, compressible barotropic flow is irrotational if every element of fluid in the flow field can be shown to have zero vorticity at some earlier time.

To be convinced that this answer is correct, consider any fluid element in the flow field. If $\boldsymbol{\Omega} = 0$ at any moment of time, then the right hand side is zero, and the value of $\boldsymbol{\Omega}/\rho$ of that particular element of fluid must remain zero. Consequently, the value of $\boldsymbol{\Omega}/\rho$ will forever remain zero for any fluid element which once upon a time had zero vorticity.

2.3 Exploiting Irrotationality

If we are looking for a flow field which is known to be irrotational, the intelligent thing to do is to introduce the scalar *velocity potential* $\phi(\mathbf{x}; t)$ by:

$$\mathbf{q} = \nabla \phi. \quad (10)$$

We have taken advantage of the irrotationality and traded in the vector variable \mathbf{q} for a single scalar variable ϕ . It is possible to derive a relatively complicated non-linear PDE for ϕ for general irrotational flows. For our purposes in this course, we will limit our attention to steady low subsonic aerodynamic flows.

3 D’Alembert Paradox

We consider the following generic aerodynamic problem: a streamlined body is placed in a steady subsonic flow. The thermodynamic state of the gas far upstream is uniform. The Reynolds number is sufficiently large that the inviscid assumption can be justified—(viscous effects are confined to thin boundary layers). The entropy distribution over the whole flow field is now a known constant! The engineering questions are: what is the lift (the component of force experienced by the body in the direction parallel to the freestream velocity vector), and what is the drag (the component perpendicular to the freestream velocity vector). What can we learn from the theory to increase the amount of lift, and/or decrease the amount of drag?

It would appear that the answers to these questions can obviously be extracted once the detailed solutions of the inviscid equations (called the Euler’s equations) have been found. A great theoretician, D’Alembert, however, discovered very early a major theoretical result which is now known as the D’Alembert Paradox. The Paradox claims that there is no inviscid drag under some very general conditions. Obviously, the conclusion is called a paradox because it is not really correct.

A non-rigorous “derivation” of the D’Alembert Paradox proceeds as follows. Consider a very large control volume enclosing the streamline body of interest, and mentally assume that the body is being held in place by a magic thread (which crosses the surface of our control volume) with tension force \mathbf{f} . Hence, \mathbf{f} is an external force acting on the control volume—in addition to fluid pressure. The momentum balance of the whole large control volume is thus:

$$\mathbf{f} - \iint_{CS} \mathbf{n}(p - p_\infty)d\mathcal{A} = \iint_{CS} (\mathbf{n} \cdot \rho\mathbf{q})\mathbf{q}d\mathcal{A}. \quad (11)$$

The left hand side is the resultant external forces acting on the control volume (the magic thread and the normal force on the surface of the control volume by the fluid static pressure). The right hand side is the net outflux of momentum crossing the surface of the control volume (we used Reynolds Transport Theorem, etc). Note that we only need the approximation that viscous stresses on the surface of the control volume be negligible—viscous forces need not be negligible inside the control volume for (11) to be valid.

The mass balance of the large control volume is:

$$0 = \iint_{CS} (\mathbf{n} \cdot \rho\mathbf{q})d\mathcal{A}. \quad (12)$$

Let's denote the freestream velocity vector by \mathbf{U}_∞ . Multiplying (12) by \mathbf{U}_∞ and subtracting the result from (11), we have:

$$\mathbf{f} = \iint_{CS} \mathbf{n}(p - p_\infty)d\mathcal{A} + \iint_{CS} (\mathbf{n} \cdot \rho\mathbf{q})(\mathbf{q} - \mathbf{U}_\infty)d\mathcal{A}. \quad (13)$$

We see that the \mathbf{f} vector, the external force required to hold the body in place, can be evaluated from detailed knowledge of the flow field on the surface of the control volume.

In essence, the D'Alembert Paradox says $\mathbf{f} = 0$. One way to determine whether D'Alembert is right or wrong is to study the two integrals on the right hand side of (13) using a large control volume.

4 Low Mach Number Aerodynamics

For steady inviscid barotropic aerodynamics, we can readily conclude that entropy and stagnation enthalpy are constant over the whole flow field. Therefore the “stagnation density” is also a constant over the whole flow field. When Mach Number M_∞ is small, we have:

$$\rho^o = \rho\left(1 + \frac{\gamma - 1}{2}M^2\right)^{1/(\gamma-1)} = \rho(1 + O(M^2)). \quad (14)$$

In other words, when $M_\infty^2 \ll 1$, $\rho \approx \text{constant}$ is a good approximation! This is the theoretical basis for making the “incompressible” approximation when the flow Mach Number is small. Almost all aerodynamics used during and before World War II used this approximation.

Once the incompressible (*i.e.* low Mach Number) approximation is (somehow) made, the continuity equation is:

$$\nabla \cdot \mathbf{q} = 0. \quad (15)$$

Instead of (10), we introduce φ as follows:

$$\mathbf{q} = U_\infty \nabla \varphi \quad (16)$$

where U_∞ is the magnitude of \mathbf{U}_∞ , and is allowed to be time dependent. Using (16) to eliminate \mathbf{q} , we obtain the spectacularly simple PDE for ϕ :

$$\nabla^2 \phi = 0. \quad (17)$$

The PDE is called the *Laplacian*, the simplest and the most studied of all second order linear PDE.

The boundary conditions for (17) are the following:

- Far away from the body of interest, we require the fluid velocity to approach the undisturbed value:

$$\lim_{\mathbf{x} \rightarrow \infty} \nabla \varphi \rightarrow \mathbf{e}_x \quad (18)$$

where \mathbf{e}_x is the unit vector in the $+x$ direction.

- On the body surface, we require the normal component of the fluid velocity to be zero:

$$(\mathbf{n} \cdot \nabla \varphi)_{\text{body surface}} = 0. \quad (19)$$

There are many methods available to solve this PDE problem. All exploits the linearity of (17), and uses the method of superposition.

For our present discussions, we shall assume the *existence* of solutions to the mathematical *boundary value problem* is not an issue. How about uniqueness? Normally, engineers do not worry too much about uniqueness. However, we shall see that uniqueness is a big deal in aerodynamics problems.

4.1 The Method of Superposition

For the moment, let us ignore the boundary conditions and concentrate on looking for solutions of (17). We assume that somehow we found N solutions, and we identify each by φ_n where the subscript is an integer index. Since each φ_n is a solution of (17), we know they all satisfy:

$$\nabla^2 \varphi_n = 0, \quad n = 1, 2, \dots, N. \quad (20)$$

It is now easy to show that

$$\phi = C_1(t)\varphi_1 + C_2(t)\varphi_2 + \dots + C_N(t)\varphi_N \quad (21)$$

also satisfies the Laplace Equation. In (21), the $C_n(t)$'s are either constants or are arbitrary functions of time.

A method of constructing solutions is thus the following. First find a collection of nice, simple “elementary” solutions. Add them together (selectively) and see what kind of flow you get.

5 The Elementary Solutions

The following is a list of simple elementary solutions of (17).

1. $\varphi_1 = ax + by + cz$, a uniform flow with velocity components a, b, c in the x, y, z direction, respectively;
2. $\varphi_2 = \frac{Q}{2\pi} \ln r$, a two-dimensional line *source or sink* at the origin;
3. $\varphi_3 = \frac{Q}{4\pi r}$, a three-dimensional point source or sink at the origin;
4. $\varphi_4 = \frac{K \cos \theta}{r}$, a two-dimensional line *doublet* at the origin;
5. $\varphi_5 = \frac{\Gamma}{2\pi} \theta$, a two-dimensional line *vortex*.

You can readily verified that these are solutions of the Laplace equation in the coordinate system indicated.

5.1 Exercises

1. Write down the Laplace equation in Cartesian Coordinates.
2. Look up in your math books for the Laplace Equation in cylindrical polar Coordinates.
3. Look up in your math books for the Laplace Equation in spherical Coordinates.
4. Show that the above listed elementary solutions indeed satisfy the respective Laplace equations.
5. Sketch, or look up in standard fluid mechanics text books, the streamlines of each of the above elementary flows.
6. Write down a solution simulating the flow over a three-dimensional egg which is oriented with the long axis aligned with the freestream flow vector.
7. Look up in standard text books or work out the elementary solution of a three-dimensional doublet.

6 The Concept of Circulation

Circulation, usually denoted by Γ and is associated with a closed loop in the flow field, is a scalar which is defined by the following line integral:

$$\Gamma \equiv \oint \mathbf{q} \cdot d\mathbf{x}. \quad (22)$$

Using the *Stokes Theorem* of vector calculus, the definition can be rewritten as:

$$\Gamma = \iint_S \mathbf{n} \cdot \boldsymbol{\Omega} dA. \quad (23)$$

where the surface of integration is any surface which uses the loop as its edge.

If we use $\mathbf{q} = \nabla\phi$ in (22), we find that the line integral can be done analytically! We obtain:

$$\Gamma = \phi_{\text{end}} - \phi_{\text{start}}. \quad (24)$$

In other words, if ϕ is assumed to be *single-valued* on the loop, Γ is zero.

6.1 Exercises

1. Which of the elementary solutions φ_n listed in the previous section is capable of being multiple valued? Note that \mathbf{q} is single-valued even when the associated φ_n is multiple valued.
2. Can you see the merits of the statement: unless the line vortex is involved, the \mathbf{f} of any aerodynamics problem of a finite body is going to be zero.
3. Ask me in class about “vortex dynamics” and about ring vortices.