

Notes #11

MAE 533, Fluid Mechanics

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1 Vorticity Theorems

We have learned that vorticity vector field $\boldsymbol{\Omega}$ is defined in terms of the velocity vector field \mathbf{q} :

$$\boldsymbol{\Omega} \equiv \nabla \times \mathbf{q}. \quad (1)$$

An immediately derivable property of $\boldsymbol{\Omega}$ is that it is divergence-free—*i.e.*

$$\nabla \cdot \boldsymbol{\Omega} = 0. \quad (2)$$

This is a mathematical property—the divergence of a curl is identically zero, always. Consider now any desired control volume (CV). We have:

$$\iint_{CS} (\mathbf{n} \cdot \boldsymbol{\Omega}) d\mathcal{A} = 0. \quad (3)$$

In English, this says the net flux of vorticity from a closed surface is zero.

We can now introduce the concept of *vortex lines*—continuous lines in space which tangent at any point is parallel to the local value of $\boldsymbol{\Omega}$ at any moment of time.¹ We can also introduce the concept of *vortex tubes*—continuous tubes in space which “walls” are all made of vortex lines.

We can now state the first vorticity theorem:

¹It is essentially the analog of a streamline for the velocity field.

- The circulation Γ of any loop lying on the surface of a vortex tube around the vortex tube (once) is a constant.

To prove this, we consider a control volume consisting of a segment of a vortex tube, and consider the vorticity fluxes from this surface. By definition of a vortex tube, $\mathbf{n} \cdot \boldsymbol{\Omega}$ is zero on the surface of the vortex tube. The two remaining surfaces are the two cross-sectional areas of the vortex tube, and their fluxes must be equal except they must have opposite signs. By the Stokes Theorem, the flux from each cross-section is the circulation of the loop which is the “edge” of the cross-sectional surface.

Here is the second Theorem:

- Vortex lines cannot end in a flow field.

Assume there is a vortex tube that ends in flow field. Consider now a control volume which lies on the surface of this vortex tube and it encloses the “end” of the tube. Now, the vorticity flux on the tube wall is zero by definition. There is now only ONE remaining surface to compute the vorticity flux. To honor (3), the circulation of the loop at this cross-section must be zero—there is no vorticity in the vortex tube. Thus, it is not possible for the circulation at this cross-section to be non-zero while the vortex tube terminates.

Here is the third Theorem:

- For inviscid barotropic flows, vorticity is convected by the fluids.

What this theorem says is that, for inviscid barotropic flows, whenever you see vorticity in a fluid element, you can be sure that it had vorticity previously. This is an obvious conclusion from the inviscid barotropic vorticity equation.

2 The Biot-Savart Law

Given a perfectly straight line vortex, the velocity field is given by the simple formula

$$v = \frac{\Gamma}{2\pi r} \quad (u = 0, w = 0), \quad (4)$$

where v is the azimuthal velocity and r is the cylindrical-polar radial coordinate (the shortest distance between the point of interest and the line vortex). What happens if you have an irrotational flow field which contains a curved

line vortex? The velocity field \mathbf{q} induced by an arbitrary (possibly curved) line vortex with (constant) circulation Γ is given by the famed Biot-Savart Law:

$$\mathbf{q} = \frac{\Gamma}{4\pi} \int \frac{\mathbf{r} \times d\mathbf{r}}{|\mathbf{r}|^3}. \quad (5)$$

Here, \mathbf{r} is the distance vector from the point of interest to a differential segment $d\mathbf{r}$ of the vortex line—the direction of $d\mathbf{r}$ is in the direction of the vortex vector (using the right hand rule). When the vortex is a straight line, this formula readily recovers (4) above.

3 Bernoulli's Equation

Under the constant density ($M_\infty^2 \ll 1$) approximation, the Bernoulli's equation for inviscid, steady flow is:

$$\frac{p}{\rho} + \frac{q^2}{2} = C \quad (6)$$

where C is a constant on any selected streamline.

For irrotational (and incompressible) flows, a Bernoulli's equation exists even for an unsteady flow field. The derivation is very simple. The inviscid momentum equation is:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = -\frac{\nabla p}{\rho}. \quad (7)$$

Using the vector identity

$$\mathbf{q} \cdot \nabla \mathbf{q} = \nabla \frac{q^2}{2} - \mathbf{q} \times \boldsymbol{\Omega} \quad (8)$$

and taking advantage that $\rho = \text{constant}$ and $\boldsymbol{\Omega} = \mathbf{0}$, we now have

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \left(\frac{p}{\rho} + \frac{q^2}{2} \right) = \mathbf{0}. \quad (9)$$

Now, we further take advantage of the existence of velocity potential for irrotational flows by substituting

$$\mathbf{q} = \nabla \phi \quad (10)$$

into (9) to obtain:

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{(\nabla \phi)^2}{2} \right) = 0. \quad (11)$$

The integral of (11) is easily obtained:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{(\nabla \phi)^2}{2} = C \quad (12)$$

where C here is a constant over the whole flow field.²

So: what are the generalizations of the inviscid, incompressible Bernoulli's equation when the flow is irrotational? The first generalization is: C is a constant over the whole flow field, and the second generalization is: there is an added term for unsteady flow.

4 Force Associated With Elementary Steady Flows

Consider a steady flow with velocity potential given by:

$$\phi = U_\infty x + \nabla \varphi. \quad (13)$$

In other words, we are considering the superposition of a uniform flow in the x -direction and some disturbance velocity potential φ . The undisturbed velocity and pressure at "infinity" are U_∞ and p_∞ , respectively. The steady Bernoulli's equation gives:

$$p - p_\infty = \rho \left(U_\infty \frac{\partial \varphi}{\partial x} + \frac{1}{2} (\nabla \varphi)^2 \right). \quad (14)$$

Equation (13) of Notes #10 now gives:

$$\mathbf{f} = \rho \iint_{CS} \left[\mathbf{n} \left(U_\infty \frac{\partial \varphi}{\partial x} + \frac{1}{2} (\nabla \varphi)^2 \right) + \mathbf{n} \cdot (\mathbf{U}_\infty + \nabla \varphi) \nabla \varphi \right] dA, \quad (15)$$

where \mathbf{f} is the external force (in addition to the pressure surface force) acting on any control volume of your choice.

²Note that C may depend on t .

Note that with the exception of the uniform flow velocity field, all the elementary solutions are “singular” at one point in space where the local velocity is infinite. Such points are called *singular points*. If we choose a control volume which does not include any singular point and we know there is no external applied force acting on the chosen control volume, then \mathbf{f} must be zero. This can readily be verified by direct analytical computations.

What happens if one chooses a control volume which includes a singular point? First of all, by the logic of the previous paragraph, we know that any control volume which includes the same singular point gives the same \mathbf{f} . But what is the value of \mathbf{f} associated with each of the elementary steady elementary flows? They are not necessarily zeros!

Let $\mathbf{F} \equiv -\mathbf{f}$, so that \mathbf{F} represents the force vector associated with the singular point (Newton’s Third Law used here). Detailed calculation will yield the following:

Source/Sink

$$\mathbf{F} = -\rho Q \mathbf{U}_\infty. \quad (16)$$

In other words, a source in a uniform flow is associated with a *thrust* and a sink is associated with a drag. This formula works for both two and three dimension flows.

Doublet

$$\mathbf{F} = 0. \quad (17)$$

In other words, a doublet in a uniform flow experiences no force at all. This formula works for both two and three dimension flows.

Line Vortex

$$\mathbf{F} = \rho \mathbf{U}_\infty \times \mathbf{\Gamma}. \quad (18)$$

In other words, a two-dimensional line vortex in a uniform flow is associated with a lift but no drag. This is called the *Joukowski Lift Law*.

All of the above results can be directly verified by analytical calculation. A useful “trick” in such calculation is to either use a very, very large control volume or a very very small control volume, taking full advantage of the fact that the answer is independent of the size and shape of the control volume, so long as it contains the singularity.

4.1 Distribution of Elementary Flows

What happens if you construct a flow with a combination of singularities? The net external force \mathbf{f} for any control volume enclosing all the singularities is then simply the sum of the respective forces associated with each of the elementary singularities.

5 The Trailing Vortex Sheet

How did we conclude that a flow is irrotational anyway? We had derived the inviscid vorticity equation previously (Notes #10, eq.(9)):

$$\frac{D}{Dt} \left(\frac{\boldsymbol{\Omega}}{\rho} \right) = \frac{\boldsymbol{\Omega}}{\rho} \cdot \nabla \mathbf{q} + \text{viscous terms.} \quad (19)$$

One of the message of this equation is: if an element of fluid had no vorticity previously, then so long as viscous effects are negligible in its journey to the present position, then its vorticity will remain zero.

In an aerodynamic problem, all the fluid elements came originally from upstream infinity where the vorticity is zero. Thus, whenever a fluid element is known to have experienced only negligible viscous effects, its vorticity will remain approximately zero. This conclusion is valid for nearly all fluid elements in the flow field, *except* for those unfortunate elements which went through the viscous boundary layer adjacent to the solid surface of the body. These fluid elements can contain vorticity, and such vorticity are to be found *downstream* of the solid obstacle in question—they are in the *wake*. Since the flow Reynolds number is very large, the boundary layers are very thin, and the *trailing vorticity wake* is essentially a *vortex sheet*.

If we now consider a control volume enclosing a finite solid body, we must deal with the fact that there may be a trailing vortex sheet intersecting the surface of your control volume, however large the chosen control volume is. In fact, if the body experiences a lift, then by Joukowski's Lift Law there must be a vortex *bounded* inside the solid body. Since vortex lines (or tubes) cannot end, they must continue on somehow. From our discussion of the vortex equation above, such vorticity can only to be found in the wake—in the trailing vortex sheet. Hence, whenever lift is generated by a finite solid body in a uniform steady flow, there must be a vorticity wake!