

Notes #11a
MAE 533, Fluid Mechanics

S. H. Lam
lam@princeton.edu
<http://www.princeton.edu/~lam>

December 8, 1998

1 The Glauert Identity

Consider the integral equation for $\gamma(x)$:

$$\frac{1}{\pi} \mathcal{P} \int_{-1}^1 \frac{\gamma(\xi) d\xi}{\xi - x} = B. \quad (1)$$

where B is a constant and the integral sign with P attached to it represents the *principal value* of the improper integral.¹

Lets change the variables as follows:

$$x = \cos \theta_*, \quad (2)$$

$$\xi = \cos \theta, \quad (3)$$

$$g(\theta) = \gamma(\xi) \sin \theta = \gamma(\xi) \sqrt{1 - \xi^2}. \quad (4)$$

The new unknown is now $g(\theta_*)$, and its integral equation is:

$$\frac{1}{\pi} \mathcal{P} \int_0^\pi \frac{g(\theta) d\theta}{\cos \theta - \cos \theta_*} = B. \quad (5)$$

¹It means an *equal* tiny gap on both sides of the singularity is omitted from the integration.

The following equation is called the Glauert Identity, and can be derived using contour integration (on the unit circle in the complex plane):

$$\frac{1}{\pi} \oint_0^\pi \frac{\cos(n\theta)d\theta}{\cos\theta - \cos\theta_*} = \frac{\sin(n\theta_*)}{\sin\theta_*}, \quad n = 0, \pm 1, \pm 2, \dots \quad (6)$$

Once we know the Glauert Identity, the solution of (5) for $g(\theta_*)$ is straightforward: just represent $g(\theta)$ by a cosine Fourier Series! So the solution of (5) is:

$$g(\theta) = A + B \cos\theta, \quad (7)$$

where A is an arbitrary constant—it is the homogeneous solution of (5). In other words, the solution of (5) is not unique; it contains an arbitrary constant A .

The solution for $\gamma(x)$ is then:

$$\gamma(x) = \frac{A}{\sin\theta_*} + \frac{B \cos\theta_*}{\sin\theta_*} = \frac{A + Bx}{\sqrt{1-x^2}}. \quad (8)$$

Note that $\gamma(x)$ is singular either at $x = -1$ or $x = +1$. We can suppress one of the two singularities by choosing A appropriately. For example, if we wish to suppress the singularity at $x = 1$, we can choose $A = -B$. We now have:

$$\gamma(x) = -B \sqrt{\frac{1-x}{1+x}}. \quad (9)$$

The constant B which originally appeared in (1) now has a physical meaning:

$$\int_{-1}^1 \gamma(x)dx = -B \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} dx = -\pi B. \quad (10)$$

In other words, it is related to the integral of γ over the span $|x| \leq 1$.

The mathematics presented in this section is very useful in the theory of thin airfoils.