

Notes #11b
MAE 533, Fluid Mechanics

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1 The Uniqueness Issue

The problem of aerodynamics of a uniform flow over a geometric solid body of given shape is to find the velocity potential ϕ as a solution of the following mathematical problem:

$$\nabla^2 \phi = 0 \tag{1}$$

The boundary conditions are:

$$(\nabla \phi)_{\mathbf{x} \rightarrow \infty} = \mathbf{U}_\infty, \tag{2}$$

$$(\mathbf{n} \cdot \nabla \phi)_{\text{body surface}} = 0. \tag{3}$$

Suppose two people solve the same problem independently, and obtain two solutions ϕ_1 and ϕ_2 . We denote the difference of the two solution by φ . The mathematical problem for φ is:

$$\nabla^2 \varphi = 0. \tag{4}$$

$$(\nabla \varphi)_{\mathbf{x} \rightarrow \infty} = 0, \tag{5}$$

$$(\mathbf{n} \cdot \nabla \varphi)_{\text{body surface}} = 0. \tag{6}$$

If we can prove that $\nabla \varphi = 0$ always, then $\nabla \phi_1 = \nabla \phi_2$ —*i.e.* the solution of the aerodynamics problem is unique.

We first tentatively assume that $\varphi \neq 0$. Multiplying (4) by φ , integrating over the whole fluid flow field and using the Divergence Theorem, we have:

$$\iiint_{CV} (\nabla\varphi)^2 dV = \iint_{CS} \varphi(\mathbf{n} \cdot \nabla\varphi) dA. \quad (7)$$

What is the control volume used? The whole fluid flow field—which excludes the solid body in question. What is the surface of this control volume? Well, it is the surface of the solid body in question, plus the surface of the “infinity” boundary of the fluid flow field. Since the Divergence Theorem requires that we have one single connected volume, we must imagine a “connection” between these two surfaces by a very thin “connector” of some kind. Where should this connector be located? The most intelligent choice would be the thin wake sheet behind the body in question—the thin sheet containing all the fluid elements which, in some previous time instant, were flowing inside the boundary layer (and therefore lost their vorticity-free guarantees).

Let us study the right-hand side of (7) on the surface of our (single) control volume. Well, on the infinity boundary, the integrand is zero—and we can show that the integrand is also zero (in spite of the huge surface area). On the solid body surface itself, the integrand is zero—and therefore the integral is zero. On any thin connector surface, we expect the value of $(\mathbf{n} \cdot \nabla\varphi)$ on opposite sides of the thin connectors to have the same value but with opposite signs. Hence, we can rewrite the surface integral there as over only half the surface area, but replacing φ by $\Delta\varphi$ which represents the difference (or “jump”) of the value of φ across the two sides of the connector surface. Now, if we assume that φ is continuous across the connector (*i.e.* $\Delta\varphi = 0$), the integration over the connector surface also vanishes. Hence, the right hand side is precisely zero. Since the integrand of the left-hand side of (7) is always positive, the only way the volume integral can be zero is: the integrand must be zero. In other words, we have succeeded in proving:

$$\nabla\varphi = 0 \quad (8)$$

when $\Delta\varphi$ is zero across the thin connector.

Look at all the elementary solutions that you have seen so far. All, except for the line vortex, are single-valued and are continuous everywhere (except at the origin). For the line vortex, however, we know that in order to make it single-valued φ must be discontinuous along a line drawn from the singularity to infinity. So the issue boils down to the following: what happens if you put

vorticities inside the solid body when you construct the solution? It is clear that if you do, you can no longer guarantee your φ to be single-valued, and that there is going to be a vortex sheet behind the solid body. It turns out that in this case, the uniqueness is not guaranteed!

1.1 The Two-Dimensional Case

Consider, for the sake of simplicity, the two dimensional case. The “connector” is now a thin channel connecting the solid body in question to the infinity surface, enclosing all the fluid elements that went through the boundary layer. What is the “jump” $\Delta\varphi$ across this thin connector channel? In general we can write:

$$\Delta\varphi = \int_a^b \nabla\varphi \cdot d\mathbf{x}. \quad (9)$$

where $\Delta\varphi = \varphi_b - \varphi_a$, and a and b represent the starting and the ending points, and the line integral is taken over the fluid irrotational flow field. If point a is immediately adjacent to point b , then the line integral is a “loop,” and $\Delta\varphi$ is essentially the *circulation* of the loop (without crossing the thin channel). For a two-dimensional irrotational flow problem, the circulation of any loop enclosing the solid body is a constant—which is not necessarily zero. How do you force uniqueness on this two-dimensional problems? You need to specify the circulation of any “irreducible” loop—such as a loop that goes around the body.

So: the message here is clear. When we look for irrotational solutions, we must be on special notice for vorticities and circulations. An aerodynamic problem is, in general, non-unique, and the non-uniqueness is related to the value of circulation(s). Eventually, we will have to deal with this troubling issue—both for two-dimensional and three dimensional problems.

1.2 Exercises

1. What do you need to solve for the irrotational aerodynamic flow field around a doughnut? a pretzel?
2. When the thickness of a vortex sheet goes to zero and $\Delta\varphi$ for a loop with points a and b immediately adjacent on top and bottom of the sheet, convince yourself that the fluid velocity field is *discontinuous* at the sheet while $\mathbf{n} \cdot \nabla\varphi$ is continuous.