

# Notes #12

## MAE 533, Fluid Mechanics

S. H. Lam  
lam@princeton.edu  
<http://www.princeton.edu/~lam>

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### 1 Theory of Thin Airfoils

We now consider the classical theory of thin airfoils. The generic problem is: we have a uniform, steady low speed ( $M_\infty^2 \ll 1$ ) flow with undisturbed velocity  $U_\infty$  in the  $+x$  direction over a thin wing with known planform and wing cross-sections located in the vicinity of the  $y \approx 0$  plane. The plan form is specified by the “chord distribution,”  $c(z)$ . The “root chord” of the wing is denoted by  $c_o = c(0)$ , the span of the wing is denoted by  $b$ . For the sake of simplicity, we shall for the moment assume that there is no “sweep.” The goal of a theory is to find out the velocity flow field about a generic thin wing, and to compute its lift and drag.

We assume the Reynolds number of the flow (based on the undisturbed incoming velocity, the root chord, and the undisturbed kinematic viscosity) is sufficiently large that the inviscid assumption outside a thin boundary layer and the trailing vortex sheet can be justified. The flow outside of the above-mentioned regions is therefore irrotational, and its velocity field  $\mathbf{q}$  can be written as:

$$\mathbf{q} = U_\infty \mathbf{x} + \nabla \varphi, \quad (1)$$

and  $\varphi$  satisfies the Laplace Equation:

$$\nabla^2 \varphi = 0, \quad (2)$$

subject to the boundary conditions that:

$$(\nabla\varphi)_{\mathbf{x}\rightarrow\infty} \rightarrow 0, \quad (3a)$$

$$(\mathbf{n} \cdot \nabla\varphi)_{\text{wing surface}} = 0. \quad (3b)$$

Taking advantage of the fact that the wing is thin so that the surface of the wing is approximately located at  $y = 0$  (on the planform), (3b) can be approximated by:

$$(\mathbf{n} \cdot \nabla\varphi)_{y=\pm 0; \text{ on planform}} = 0. \quad (4)$$

The use of (4) instead of (3b) is called the mean-surface approximation.

Equation (3b) or (4) says that the fluid velocity vector is a tangent vector of the surface of the solid body. We can express this boundary condition as follows:

$$\left( \frac{\frac{\partial\varphi}{\partial y}}{U_\infty + \frac{\partial\varphi}{\partial x}} \right)_{y=\pm 0; \text{ on planform}} = \frac{\partial Y_\pm(x, z)}{\partial x} \quad (5)$$

Neglecting the perturbation velocity in comparison to  $U_\infty$  in the denominator of the left hand side, we have, finally:

$$\frac{1}{U_\infty} \left( \frac{\partial\varphi}{\partial y} \right)_{y=\pm 0; \text{ on planform}} = \frac{\partial Y_\pm(x, z)}{\partial x}. \quad (6)$$

## 1.1 Problem Decomposition

Let the upper and lower surfaces of the thin wing be specified by  $y_\pm = Y_\pm(x, z)$ , respectively. The *camber* distribution,  $Y_{\text{camber}}(x, z)$ , and the *thickness* distribution,  $Y_{\text{thick}}(x, z)$ , are defined as follows:

$$Y_{\text{camber}}(x, z) = \frac{Y_+(x, z) + Y_-(x, z)}{2}, \quad (7)$$

$$Y_{\text{thick}}(x, z) = \frac{Y_+(x, z) - Y_-(x, z)}{2}. \quad (8)$$

We can readily express  $Y_\pm$  in terms of  $Y_{\text{camber}}$  and  $Y_{\text{thick}}$ :

$$Y_\pm = Y_{\text{camber}} \pm Y_{\text{thick}}. \quad (9)$$

We can now rewrite (6) as follows:

$$\frac{1}{U_\infty} \left( \frac{\partial\varphi}{\partial y} \right)_{y=\pm 0; \text{ on planform}} = \frac{\partial Y_{\text{camber}}}{\partial x} \pm \frac{\partial Y_{\text{thick}}(x, z)}{\partial x}. \quad (10)$$

We now reap the benefits of the mean-surface approximation: any thin wing problem can be decomposed into the superposition of a *camber problem* (a wing with zero thickness) and a *thickness problem* (a wing with zero camber). In other words, we can decompose  $\varphi$  by:

$$\varphi = \varphi_a + \varphi_s \tag{11}$$

where subscripts  $a$  and  $s$  are short-hand for “anti-symmetric” and “symmetric,” respectively. The boundary conditions on the “body” for  $\varphi_a$  and  $\varphi_s$  are:

$$\frac{1}{U_\infty} \left( \frac{\partial \varphi_a}{\partial y} \right)_{y=\pm 0; \text{ on planform}} = \frac{\partial Y_{\text{camber}}}{\partial x}, \tag{12}$$

$$\frac{1}{U_\infty} \left( \frac{\partial \varphi_s}{\partial y} \right)_{y=\pm 0; \text{ on planform}} = \pm \frac{\partial Y_{\text{thick}}(x, z)}{\partial x} \tag{13}$$

It is easy to be convince that he pressure field of the camber problem is anti-symmetric (with respect to the  $y = 0$  surface), while it is symmetric for the thickness problem.

Most importantly, the solutions of the camber and the thickness problems are totally uncoupled and are independent of each other.<sup>1</sup>

## 1.2 The Thickness Problem

Physically, the thickness problem deals with a wing with a symmetric (with respect to  $y = 0$ ) cross-section profile at zero angle of attack. Hence,  $\varphi_s$  can be constructed by a distribution of sources (or sinks). The general strategy for numerical solution should be clear: cut up the planform into  $N$  tiny areas, and put in each tiny area a (smeared) source of unknown strength; the  $N$  source strengths are to be determined by the requirements that  $N$  selected points on the wing surface satisfy the boundary conditions. In fact, the same general strategy can be used analytically, replacing the  $N$  unknown source strengths by an unknown “source distribution.”

What do we get at the end of all the hard work? We get the contribution to pressure in the flow field by the thickness problem. How much lift does this solution generate (assuming the body is closed *i.e.* the net source is zero)? The answer is zero (and it is obvious). How much drag does this

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<sup>1</sup>This would not be so if we did not make the mean-surface approximation.

solution generate? The answer is again zero (and it is obvious with some thoughts).

The answer obtained from this part of the solution is of interest mainly to the structural engineers who need to design the trusses for the wing and the boundary layer people who are interested in transition Reynolds numbers, separations and other boundary-layer related issues.

### 1.3 The Camber Problem

Physically, the camber problem deals with a wing with zero thickness which can vary its angle of attack. Intuitively, we expect lift (and perhaps even drag) for this class of problems.

To avoid the complications of the presence of a trailing vortex wake, we shall first limit ourselves to two-dimensional problems. The solution for strictly two-dimensional anti-symmetric flows can be constructed using a line vortex distribution  $\gamma(x)$  (circulation per unit chord length). You may recall that I have repeatedly warned about the issue of uniqueness of solutions.

The boundary condition on the mean-surface is:

$$G(x) = \frac{1}{2\pi} \int_{-c_o/2}^{c_o/2} \frac{\gamma(\xi) d\xi}{x - \xi}. \quad (14)$$

where the chord of the wing is  $c_o$  and  $g(x)$  is solely determined by the geometry of the camber line:

$$G(x) \equiv \frac{dY_{camber}}{dx} = \frac{1}{U_\infty} \left( \frac{\partial \varphi_a}{\partial y} \right)_{y=0}. \quad (15)$$

Note that the improper integral with the  $P$ -overstrike in (14) indicate *principal value* is to be used. Equation (14) is a linear integral equation for the unknown vortex distribution  $\gamma(x)$ .

It is customary to change variables from  $\xi$  and  $x$  to  $\theta$  and  $\theta_*$ :

$$x = \frac{c_o}{2} \cos \theta_*, \quad (16)$$

$$\xi = \frac{c_o}{2} \cos \theta, \quad (17)$$

$$g(\theta) = \frac{\gamma(\theta) \sin \theta}{2}. \quad (18)$$

Equation (14) becomes:

$$G(\theta_*) = \frac{1}{\pi} \int_0^\pi \frac{g(\theta) d\theta}{\cos \theta_* - \cos \theta}. \quad (19)$$

Now,  $G(\theta_*)$  is a known function while  $g(\theta)$  is the unknown to be solved for from this linear integral equation.

The solution method exploits the Glauert Identity. The unknown function  $g(\theta)$  is represented by a Fourier Cosine Series:

$$g(\theta) = \sum_{n=0}^{\infty} A_n \cos n\theta, \quad (20)$$

and the known function  $G(\theta_*) \sin \theta_*$  is represented by a Sine Fourier Series:

$$G(\theta_*) \sin \theta_* = \sum_{n=0}^{\infty} B_n \sin n\theta_*. \quad (21)$$

Given the geometry of the camber line, the constants  $B_n$ 's are known numbers. It is then a straightforward matter to determine the  $A_n$ 's in terms of the  $B_n$ 's.

Except, of course, the value of  $A_0$ . Yes, any  $A_0$  will honor the integral equation being solved. Yes, the solution to the camber problem is now explicitly shown to be non-unique!

### 1.3.1 The Kutta Condition

The solution of the camber problem generally exhibits “singular” behavior at both the leading edge and the trailing edge. Kutta was the first fluid mechanicist to confront the non-uniqueness issue. He proposed to use the undetermined  $A_0$  to eliminate the singular behavior at the trailing edge, under the assumption that the thin wings in question all have sharp trailing edges while their leading edges are rounded and are smooth.

The rationale for the Kutta condition is simple: the unique solutions so generated are in good agreement with experiments.

## 1.4 Exercises

1. Do the flat plate (at angle of attack  $\alpha$ ) problem.
2. Do the simplest camber airfoil problem (only  $B_2 \neq 0$ ).