Notes #13
MAE 533, Fluid Mechanics

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1 Prandtl's Theory of Finte Wing

Let (x, y, z) be a rectangular coordinate system, with $+x$ in the direction of the undisturbed freestream flow velocity, $+y$ normal to the plane of the thin wing, and z in the direction of the "span" of the wing. The wing is located on the $y = 0$ plane, "centered" at $z = 0$. We are given a large aspect ratio wing with the following information:

Planform of the wing is described by $c(z)$, the chord distribution. The "root chord," $c(0)$, is denoted by c_o. The span of the wing (distance from wing tip to wing tip), is denoted by b. A large aspect ratio wing means $b >> c_o$.

Twist of the wing is described by $\alpha_g(z)$, the distribution of the zero-lift angle of the wing (due to the camber of the wing cross-section).

The overall *angle of attack* of the wing shall be denoted by α_w. The goal of Prandtl's theory of finite wing is to find the lift and drag coefficients of this wing for low speed flight ($M_\infty^2 << 1$) as a function of the above information. There is no sweep to the wing under consideration.

1.1 The Trailing Vortex Wake

Let the lift per unit span of the wing be denoted by $\ell(z)$. Taking advantage of the large aspect ratio assumption, it is concluded that the local aerodynamics (at any z station) can be approximated by two-dimensional theory.

- Using the Joukowsky Lift Law, we have:

$$\ell(z) = \rho U_\infty \Gamma(z) \tag{1}$$

where $\Gamma(z)$ is the *circulation distribution* over the span of the wing.

- Using two-dimensional theory of thin wings, $\ell(z)$ can also be expressed as follows:

$$\ell(z) = \frac{C_\ell}{2} \rho U_\infty^2 c(z) \alpha_{\text{eff}}(z) \tag{2}$$

where $C_\ell = 2\pi$ and $\alpha_{\text{eff}}(z)$ is the "effective" angle of attack.

- If there were no trailing vortex wake, we would have:

$$\alpha_{\text{eff}}(z) = \alpha_w + \alpha_g(z) \tag{3}$$

which is derived by merely looking at the geometry of the problem.

- But there is a trailing vortex wake! The trailing vortex wake will induce a "downwash" at the (quarter chord) of the large aspect ratio wing. This downwash rotates the incoming flow velocity vector *downward* by an angle $\alpha_i(z)$. Hence, we have:

$$\alpha_{\text{eff}}(z) = \alpha_w + \alpha_g(z) - \alpha_i(z). \tag{4}$$

- Equating (1) and (2), we obtain:

$$\Gamma = \pi U_\infty c(z) \left(\alpha_w + \alpha_g(z) - \alpha_i(z)\right). \tag{5}$$

We now need a relation between $\alpha_i(z)$ and $Gamma(z)$.

1.2 The Integro-differential Equation for $Gamma(z)$

What is the strength of the trailing vortex from a span station located at $z = \eta$? We all remember that the divergence of vorticity is zero. In plain English, it says the net flux of vorticity across a closed surface is zero. Using the Stokes Theorem, this can be translated to say the net circulation crossing a closed surface is zero. Hence, Γ were a constant at that station, there would be no trailing vortex—whatever circulation comes in at station $z = \eta$ comes out at station $z = \eta + d\eta$. What happens if $d\Gamma/dz \neq 0$? the circulation of the trailing vortex issued between $z = \eta$ and $z = \eta + d\eta$ must be $(d\Gamma/d\eta)d\eta$ (the sign of this term depends on the arrow on the picture representing the vorticity). What is the contribution to the induced downwash v_{dw} at station z due to a trailing vortex issued at station η? Taking half of the induced velocity of a doubly infinite line vortex, we have:

$$d(v_{dw}) = \frac{1}{4\pi}\frac{(d\Gamma/d\eta)d\eta}{\eta - z} \tag{6}$$

The approximation that the trailing vorticity are rectilinear had been made. The sum of all the trailing vorticities in the vortex wake is therefore:

$$v_{dw}(z) = \frac{1}{4\pi}\,⨍_{-b/2}^{b/2} \frac{(d\Gamma/d\eta)d\eta}{\eta - z} \tag{7}$$

Geometrically, we have:

$$\alpha_i(z) = \frac{v_{dw}(z)}{U_\infty}. \tag{8}$$

Thus the final *integro-differential equation* for the unknown $\Gamma(z)$ is:

$$\frac{\Gamma}{\pi U_\infty c(z)} = \alpha_w + \alpha_g(z) - \frac{1}{4\pi U_\infty}\,⨍_{-b/2}^{b/2} \frac{(d\Gamma/d\eta)d\eta}{\eta - z}. \tag{9}$$

The remaining task is to find $\Gamma(z)$ as a function of the parameters of the wing: $c(z)$, b, α_w and $\alpha_g(z)$. You should be able to see Professor Glauert smiling at this equation.

Note that (9) is a linear equation, and α_w and $\alpha_g(z)$ are "forcing terms." The principle of superposition can be used, and the impacts of each of these terms can be computed separately.

It is of interest to point out that the solution to (9) is unique. This is in contrast to the two-dimensional camber thin wing problem which was found to be non-unique, and had to bring in the Kutta condition to rationale the final choice of the solution there.

1.3 The Transformation

We introduce θ_* and θ as was done previously:

$$
\begin{aligned}
z &= \frac{b}{2}\cos\theta_*, && (10)\\
\eta &= \frac{b}{2}\cos\theta. && (11)
\end{aligned}
$$

We further introduce new variables $G(\theta_*)$ and $C(\theta_*)$:

$$
\begin{aligned}
\Gamma(z) &= U_\infty c_o G(\theta_*), && (12)\\
c(z) &= c_o \sin\theta_* C(\theta_*). && (13)
\end{aligned}
$$

Using straightforward manipulations, we have:

$$
\frac{1}{\pi}\frac{G}{C(\theta_*)\sin\theta_*} = \alpha_w + \alpha_g(\theta_*) - \frac{c_o}{2\pi b} P\!\!\!\!\!\!\int_0^\pi \frac{(dG/d\theta)d\theta}{\cos\theta - \cos\theta_*} \qquad (14)
$$

Before we go to find the A_n, lets first compute the total lift and induced drag of the whole wing:

$$
\begin{aligned}
L &= \int_{-b/2}^{b/2} \ell(z)dz = \frac{1}{2}\rho U_\infty^2 b c_o \int_0^\pi G(\theta)\sin\theta d\theta, && (15)\\
D_i &= \int_{-b/2}^{b/2} \ell(z)\alpha_i(z)dz = \frac{1}{2}\rho U_\infty^2 b c_o \int_0^\pi G(\theta)\alpha_i(\theta)\sin\theta d\theta. && (16)
\end{aligned}
$$

Where did the induced drag come from? It comes from the *rotation* of the lift vector by the downwash!

We now use a Fouier Sine Series to represent our unknown $G(\theta)$:

$$
\begin{aligned}
G(\theta) &= \sum_{n=0}^{\infty} A_n \sin n\theta, && (17)\\
\frac{dG}{d\theta} &= \sum_{n=0}^{\infty} nA_n \cos n\theta. && (18)
\end{aligned}
$$

It is immediately obvious from (15) that A_1 is the only coefficient that contributes to the wing lift. Using the Glauert Identify, we have:

$$
\alpha_i(\theta_*) = \frac{c_o}{2b}\sum_{n=0}^{\infty}\frac{nA_n\sin n\theta_*}{\sin\theta_*}. \qquad (19)
$$

The equation for the Fourier coefficients is simply:

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{A_n \sin n\theta_*}{C(\theta_*)\sin\theta_*} = \alpha_w + \alpha_g(\theta_*) - \frac{c_o}{2b}\sum_{n=0}^{\infty}\frac{nA_n \sin n\theta_*}{\sin\theta_*}. \tag{20}$$

The total wing lift and induced drag, (15) and (16), can now be rewritten as follows:

$$L = \frac{\pi}{4}\rho U_\infty^2 b c_o A_1, \tag{21}$$

$$D_i = \frac{\pi}{4}\rho U_\infty^2 b c_o \left(\frac{c_o}{2b}\right)\sum_{n=0}^{\infty} nA_n^2 \tag{22}$$

It is interesting to note that D is always positive (it is never a thrust!). Note that D for fixed L, D is inversely proportional to b. In plain English: long wing span gives you less drag.

It is important to note that the drag calculated is inviscid drag, and its mechanicsm is totally vorticity based. Note also that it is possible to have a wing with no net lift, yet it has a net inviscid drag. All it needs is a vortex wake.

1.4 The Elliptic Wing Solution

For a given wing, one can use (20) to compute for as many A_n's as your patience would endure. For example: when 100 terms are desired for the Fourier Sine Series for G, pick 100 point on the span of the wing to make (20) happy. Its only a matter of programming and computations.

The case of an elliptical wing is represented by $C(\theta_*) = 1$, and the particular solution associated with a wing with no twist can be obtain analytically. The area S of an ellipse with b and c_o as the major and minor axis is $\pi b c_o/4$. For this case, forcing term of (20) is simply a constant. Hence, only A_1 is non-zero. Doing the algebra, we obtain

$$A_1 = \frac{\pi\alpha_w}{1 + \frac{\pi c_o}{2b}}. \tag{23}$$

and the wing lift coefficient can be obtained from (15) as:

$$C_L = \text{wing lift coefficient} \equiv \frac{L}{\frac{1}{2}\rho U_\infty^2 S} = 2A_1 = \frac{2\pi\alpha_w}{1 + \frac{\pi c_o}{2b}} \tag{24}$$

We see that in the $c_o/b \to 0$ limit of two dimensional flows, the lift curve slope recovers the classical value of 2π. Hence, for a finite wing, the lift curve slope is always smaller than the two-dimensional value.

For this case, the induced drag coefficient is readily computed:

$$C_{D_i} = \text{induced drag coefficient} \equiv \frac{D_i}{\frac{1}{2}\rho U_\infty^2 S} = \left(\frac{c_o}{8b}\right) C_L^2 \tag{25}$$

The derivation that the induced drag coefficient of a finite elliptic wing is proportional to the square of its lift coefficient is a major triumph of theoretical aerodynamics. The bigger the span in comparison to the chord, the lower the induced drag. It is a fundamental idea in aircraft design.

2 The Total Drag

The total drag D experienced by an aircraft is the sum of the inviscid induced drag D_i and the viscous drag D_v caused by the boundary layers. We have:

$$D = D_v + D_i. \tag{26}$$

The viscous drag can be expressed in terms of a viscous drag coefficient:

$$D_v = C_{D_v} \frac{1}{2} \rho U_\infty^2 S, \tag{27}$$

where C_{D_v} is a small dimensionless number which depends (weakly) on Reynolds number in the practical range of Reynolds number of interest.

The obviously interesting question is: which drag is more important?

2.1 Exercises

- Can you figure out how to do the $\alpha_g(z) \neq 0$ problem?
- Watch for my lecture, and understand the rationalizations which shows that $D_v \approx D_i$ in the cruising mode of a transport aircraft.