

Notes #2  
MAE 533, Fluid Mechanics  
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I am writing these notes as the semester proceeds. I am most likely to update them from time to time. Please destroy the old version whenever a new version becomes available. Do not distribute these notes without my permission.

## 1 Dependent and Independent Variables

What are the dependent variables in a fluid mechanics problem? The following list should be obvious: density  $\rho$ , velocity vector  $\mathbf{q}$ , pressure  $p$ , temperature  $T$ . And perhaps others.

Now do we know what these words mean? What is density? Density is mass per unit volume, and is a scalar! How would you measure it, conceptually? You magically scoop a tiny volume of the stuff, measure its mass and volume, and do a division. We agree that density is a state variable (do you know the definition of a state variable?). So far, so good. (question: how tiny should the tiny volume be?)

What is velocity (vector)  $\mathbf{q}$ ? Well, it is a three-dimensional vector in our Euclidean Space (do you know what that is?). How do you measure it, conceptually? Well, you magically spray a tiny volume of the stuff with massless pink paint, and take two snapshots of the fluid with your camera

pointing in the  $+z$  direction, one after the other  $\Delta t$  seconds apart. It is then obvious that you can find the  $x$  and  $y$  components of  $\mathbf{q}$  from the resulting photos . . . . (questions: how tiny should the sprayed glob be, and how small should  $\Delta t$  be?)

What is pressure? Well, it is a scalar (it is a number with no direction). What is its unit? It is force per unit area exerted by the fluid. Now, force is a vector, and pressure is a scalar, so obviously we need to modify what we just said. A more precise definition is: it is the “normal component” of force acting on an element of area divided by the area exerted by the fluid. Now, how do we know that pressure so defined is independent of the orientation (i.e. direction) of the surface element? Our common daily experience tells us it is so when the fluid is not moving. What happens when the fluid moves? We shall see later that it is no longer so. For the moment, we will not deal with this subtle point. We will tentatively agree that pressure is the algebraic average of three (mutually orthogonal) normal force per unit areas. In addition, we will tentatively agree that it is also a *state variable*. We will worry about the impreciseness later.

What is temperature? We have lots of vagueness here. Let's agree to worry about the impreciseness for the moment, and also agree that it is a state variable.

We can go on to discuss other dependent variables. The point that I am trying to make is: one must always have a clear definition (or recognize the vagueness) of words and terms that one talks about.

The independent variables of the problem is, in general, the vector  $\mathbf{x}$  and the scalar  $t$ .

## 2 Conservation Laws

We consider a flow field in physical space with  $\rho(\mathbf{x}, t)$ ,  $\mathbf{q}(\mathbf{x}, t)$ ,  $p(\mathbf{x}, t)$ ,  $T(\mathbf{x}, t)$  etc. Obviously, they together must satisfy the Laws of Nature.

### 2.1 Conservation of Mass

The Law of Conservation of Mass is well accepted (in classical physics). In English, it says “mass cannot be created or destroyed.” How do we say it in mathematics?

Here it goes. We mentally choose an *arbitrary* finite glob of fluid at time  $t$ , and mentally spray it pink. The pink stuff is our apple—which is normally called the “system” by all. What is the mass  $m$  of the system? It is, by our definition of  $\rho$ :

$$m(t) = \iiint_{\text{SysVol}} \rho(\mathbf{x}, \mathbf{t}) dV. \quad (1)$$

Hence, the conservation of mass says that  $m$  will not change (with time). We have:

$$\frac{dm}{dt} = 0. \quad (2)$$

In English, (2) says: the time rate of change of the mass of the system is zero. A good paraphrase of what we know as the conservation of mass.

Substituting (1) into (2), we have:

$$\frac{d}{dt} \iiint_{\text{SysVol}} \rho(\mathbf{x}, \mathbf{t}) dV = 0. \quad (3)$$

Now, we would like to, somehow, move the time derivative operator inside the triple volume integration. It is clear that we can't just simple-mindedly move it inside because the limits of integration of the volume integral is time dependent when the fluid is in motion. The procedure to deal with taking a time derivative of a volume integral of a moving system is something you must master. It is known in the fluid mechanics community as the Reynolds Transport Theorem.

The idea of the procedure is very simple: one simply goes back to the definition of derivative. Equation (3) can be rewritten as:

$$\lim_{\Delta t \rightarrow 0} \frac{m(t + \Delta t) - m(t)}{\Delta t} = 0. \quad (4)$$

Watch for the presentation in class for the details. When all the dusts settled, we have:

$$\frac{d}{dt} \iiint_{\text{SysVol}} \rho(\mathbf{x}, \mathbf{t}) dV = \iiint_{\text{SysVol}} \left\{ \frac{\partial \rho}{\partial t} \right\} dV + \iint_{\text{SysSurf}} \{ \mathbf{n} \cdot (\rho \mathbf{q}) \} dA \quad (5)$$

where  $\mathbf{n}$  is the *unit outward normal* of the surface element  $dA$ . Equation (5) is called the Reynolds Transport Theorem, and  $\rho(\mathbf{x}, \mathbf{t})$  can be any scalar.

Using the appropriate divergence theorem in Notes #1 to convert the surface integral to a volume integral, we have:

$$\frac{dm}{dt} = \iiint_{\text{SysVol}} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \right\} dV = 0. \quad (6)$$

Now, since the volume integral is zero for any *arbitrarily* chosen system volume, the integrand must be zero. Hence, we have arrived at the differential form of the Law of Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = \mathbf{0}. \quad (7)$$

This equation is written in vector form. You should know how to write it in any common coordinate system (rectangular, polar, and where to look it up for spherical).

Working out the divergence of the product  $\rho \mathbf{q}$ , we obtain from (7) an useful alternative form:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{q} = \mathbf{0}. \quad (8)$$

where the operator  $D/Dt$  is called the *substantial derivative* (or *material derivative*, or *Lagrangian derivative*), defined by:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla. \quad (9)$$

The physical meaning of the substantial derivative is: it is the time rate of change of something about an identified glob of fluid (your system) when you follow the glob of fluid as it moves. In (8), it is being applied to a scalar. Warning: extreme care must be exercised when it is to be applied to a vector, as we shall see later.

The validity of (7) and (8) is quite general.

## 2.2 The Concept of Flux

Let  $\varphi(\mathbf{x}, \mathbf{t})$  denote “some stuff” per unit volume of fluid. For example,  $\varphi = \rho$  is mass per unit volume, and  $\varphi = \rho e$  is internal energy per unit volume (and  $e$  is internal energy per unit mass). If there is dust particules in the air,  $\varphi$  could represent the number of dust particules per cc.

We now consider an element of surface area  $dA$  in the flow field. Its unit normal is denoted by  $\mathbf{n}$ . Now, how much “stuff” flows across the surface

element in the general direction of  $\mathbf{n}$  per unit time? The amount is called *flux*. It is easy to be convinced that:

$$\varphi\text{-flux contributed by } dA = \{\mathbf{n} \cdot (\varphi\mathbf{q})\} dA. \quad (10)$$

Hence, the total amount of “outflux” of  $\varphi$  from any closed surface is the surface integral. This provides a physical interpretation of the second term on the right hand side of (5), the Reynolds Transport Theorem.

### 3 The Concept of Control Volume

Let us now consider a *control volume* which is fixed in space (not moving) and does not change its shape in time. We choose it to be *identical* to the system volume at time  $t$ .

The Law of conservation of mass can now be written in terms of the control volume (C.V.) as follows:

$$\iint_{\text{C.V.}} \{\mathbf{n} \cdot (\rho\mathbf{q})\} dA = -\frac{dm_{\text{C.V.}}}{dt}. \quad (11)$$

where  $m_{\text{C.V.}}$  is the mass of the fluid inside the fixed control volume:

$$m_{\text{C.V.}} \equiv \iiint_{\text{C.V.}} \rho(\mathbf{x}, \mathbf{t}) dV. \quad (12)$$

We can now express in English the Law of Conservation of Mass using the concept of control volume:

**The net mass outflux from a fixed control volume equals  
to the mass loss of the control volume.**

This is the same fundamental idea as “mass cannot be created or destroyed.” But it is being expressed in terms of a fixed control volume.

#### 3.1 Homework

Watch the course web page. I may post homework excercises later in the week.

Think about how you would formulate the linear momentum equation and/or the energy equation.