

Notes #2a
MAE 533, Fluid Mechanics
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1 Index Notation

The *index notation* is a very useful tool in the study of continuum mechanics and many other fields.

Usually, the components of a vector are denoted by distinct letters. For example, the Cartesian components of the position vector \mathbf{x} are usually denoted by x , y and z . We shall find it convenient to denote them x_1 , x_2 and x_3 instead. Similarly, the Cartesian components of the velocity vector \mathbf{q} shall be denoted by q_1 , q_2 and q_3 instead of the conventional u , v and w .

The continuity equation of fluid mechanics is, in conventional vector notation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \quad (1)$$

In cartesian coordinates, the conventional form of the above is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \quad (2)$$

In index notation, this becomes:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^3 \frac{\partial \rho q_i}{\partial x_i} = 0 \quad (3)$$

The great Einstein proposed the so-called Einstein summation convention:

Whenever a term contains a repeated index, a summation with respect to that index is assumed.

Hence, we have for our continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho q_i}{\partial x_i} = 0 \quad (4)$$

where the summation sign for the second term has been omitted.

Given two vectors \mathbf{a} and \mathbf{b} , the inner (dot) product can now be written as:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i.$$

1.1 The Cross Product and the Curl

Given two vectors \mathbf{a} and \mathbf{b} , the cross product $\mathbf{a} \times \mathbf{b}$ can be obtained from the following “matrix” formula:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (5)$$

where the elements in the top row are unit vectors in the $+x$, $+y$ and $+z$ directions. In index notation, the cross product of \mathbf{a} with \mathbf{b} is written as:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k \quad (6)$$

and the summation convention (on the repeated indices ‘j’ and ‘k’ are in force.

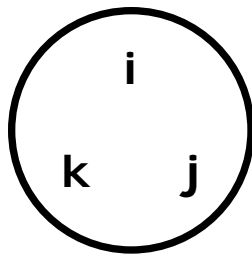


Figure 1: The “Clock” of the Permutation Matrix.

The ϵ_{ijk} is a permutation matrix, defined by:

1. $\epsilon_{ijk} = 0$ whenever any index is repeated. For example, $\epsilon_{112} = 0$, $\epsilon_{323} = 0$;
2. $\epsilon_{ijk} = +1$ whenever ijk is increasing in the clockwise direction;
3. $\epsilon_{ijk} = -1$ whenever ijk is increasing in the counter-clockwise direction.

Note that $\epsilon_{ijk} = -\epsilon_{ikj}$ always—the sign is changed whenever two indices switch places. You can readily verify that (6) is correct by doing the j and k summation.

I assume everybody knows how to find the Curl of a vector field. In Cartesian coordinates, it can also be obtained from the following matrix formula:

$$\nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}. \quad (7)$$

The Curl of \mathbf{q} is written as:

$$(\nabla \times \mathbf{q})_i = \epsilon_{ijk} \frac{\partial q_k}{\partial x_j} \quad (8)$$

and the summation convention (on the repeated indices 'j' and 'k') are in force.

The most important identity involving permutation matrices is:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \quad (9)$$

where δ_{mn} is called the “Kronecker Delta,” which is defined by $\delta_{mn} = 1$ if $m = n$, otherwise it equals to zero. Note that the repeated index of the left hand side is ‘i’.

1.2 Exercises

1. Let \mathbf{a} , \mathbf{b} and \mathbf{c} denote three vectors. Write down $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ in index notation.
2. Write down the substantial derivative D/Dt in index notation.
3. Show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

4. Show that

$$\mathbf{q} \cdot \nabla \mathbf{q} = \frac{1}{2} \nabla q^2 - \mathbf{q} \times (\nabla \times \mathbf{q}).$$

This identity is most important in fluid mechanics. Hint: start from the second term on the right hand side.

5. If \mathbf{a} and \mathbf{b} are expressed in cylindrical coordinates (r, θ, z) , what is the matrix form of $\mathbf{a} \times \mathbf{b}$ and $\nabla \times \mathbf{a}$? (look them up).

2 The Stress Tensor Π

Newton, after examining data from experiments with common fluids, concluded that the x component of the *shear stress* (force per unit area) on a flat surface ($y = 0$) is $\mu \partial u / \partial y$, where μ , a property of the fluid, is called the viscosity of the fluid.

The use of the index notation makes the dealings of the stress tensor Π much easier. Let Π_{ij} denote the j -component of stress (the j component of the force vector per unit area) on an element of surface whose unit normal \mathbf{n} is in the $+i$ direction. Newton's experimental observations can then be written as:

$$\Pi_{21} = \mu \frac{\partial q_1}{\partial x_2}.$$

The question now is: what is the "most general" form for Π_{ij} which will recover Newton's observation?

There are some common sense constraints:

1. Π_{ij} must be zero if \mathbf{q} is a constant vector. Hence, the value of its elements must depend on some non-uniformity of the flow field.
2. Fluids such as ordinary gas and liquid has no preferred directions. In other words, we shall assume that they are "isotropic."
3. $\Pi_{ij} = \Pi_{ji}$; in other words, the matrix is symmetric. We shall discuss this in class.

Prompted by item #1, Navier and Stokes assumed that Π_{ij} depends linearly on $\partial q_i / \partial x_j$. They proposed:

$$\Pi_{ij} = \mu \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial q_k}{\partial x_k}, \quad (10)$$

where μ and λ are material properties. You can easily verify that this formula will indeed recover Newton's formula (and μ is identified as the fluid viscosity). So, if one wishes to measure μ for a given fluid, one needs only to perform Newton's experiment. In general, $\mu > 0$ (the positive definiteness is required by the Second Law of Thermodynamics). What happens at a solid-fluid interface?

But what about λ ? How does one go about measuring it?

2.1 Viscous Normal Stress

What is pressure? Well, we agreed earlier that pressure is the normal force per unit area—when the fluid is at rest. What happens when the fluid is moving (with non-uniform velocity)?

This Navier-Stokes stress tensor says non-zero normal stress Π_{ii} will result when the viscous flow field is non-uniform. In general, Π_{11} and Π_{22} and Π_{33} are *NOT* the same! So, if you were to somehow measure the normal force per unit area in a non-uniform viscous flow field, your answer will depend on the orientation of your element of measuring area. If this is indeed so, that what is the “meaning” of pressure when the viscous fluid is in non-uniform motion?

Stokes thought about this problem, and proposed the following “postulate:”

Pressure at a given point is the algebraic average of three mutually orthogonal normal stresses measured about that point.

This totally sensible postulate enables us to determine λ . You can easily verify that the Stokes Postulate yields

$$\lambda = -\frac{2\mu}{3}.$$

Experimentally, this theoretical result is obeyed for simple gases. The value of $\lambda + 2\mu/3$ is often called the second coefficient of viscosity, and its theoretical basis is not completely resolved.

2.2 Kinematic Viscosity

Kinematic viscosity ν is defined by:

$$\nu \equiv \frac{\mu}{\rho}$$

2.3 No-Slip Condition

It is observed experimentally that the fluid “sticks” to the solid. This is called the “*no-slip condition*.”

2.4 Exercises

1. What is the dimension of μ , in terms of M (mass), ℓ (length), and t (time).
2. Pete Sampras’s serve has been clocked at about 125 miles per hour. And you know approximately the diameter of a tennis ball. Estimate the Reynolds number of the flow about this tennis ball in air and in water. How “slow” must this tennis ball move if you want its Reynolds number to be of order unity?
3. Non-dimensionalize our variables so that the non-dimensionalized variables are expected to be of order unity. And we shall mark all non-dimensionalized variables by an asterisk. So, we have

$$x_i^* = \frac{x_i}{\ell}, \quad q_i^* = \frac{q_i}{U_o}$$

where ℓ and U_o are the characteristic length and velocity of the problem. If the problem is time-dependent, let τ represent the characteristic time scale of the time-dependency, and

$$t^* = \frac{t}{\tau}$$

. For dependent variables, we can choose

$$\rho^* = \frac{\rho}{\rho_o}, \quad p^* = \frac{p - p_o}{\rho U_o^2}$$

where ρ_o and p_o are characteristic values for ρ and p , respectively.

- Write down the continuity and momentum equations in non-dimensional form. How many dimensionless variables do you find? What are they?
- When can you make the “quasi-steady” approximation (neglect the $\partial/\partial t$ terms)?

- When can you make the “inviscid” approximation (neglect the viscous stress terms)?
- Get the dimensionless energy equation. You are on your own to do the needed non-dimensionalizations. The Prandtl number P_r is defined by:

$$P_r \equiv \frac{\mu C_p}{k}$$

where C_p is the specific heat at constant pressure. Can you make it appear in your energy equation?