

Notes #3

MAE 533, Fluid Mechanics

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1 Comments on the Navier Stokes Equations

When the “Fourier Law of Heat Conduction” for heat conduction in the fluid (*i.e.* the heat flux vector—energy per unit area per unit time—is $-k\nabla T$ where k is heat conductivity and T is temperature) and the symmetric stress tensor Π_{ij} given in Notes #2 are adopted, the resulting PDEs are referred to as the *Navier-Stokes Equations*. Three material property coefficients are needed: the Fourier heat conductivity coefficient k , the viscosity coefficient μ , and the second coefficient of viscosity, often denoted by λ . These are all “phenomenological coefficients,” and are to be measured experimentally and recorded in handbooks (as functions of temperature and density) for all to use. You should know how to find these numbers from your favorite text books or handbooks (what happens to μ when temperature increases or density increases?).

The Navier-Stokes equations need to be supplemented by an “equation of state” under the assumption that the fluid is in thermodynamic equilibrium. For example, the well known perfect gas law $p = \rho RT$ is an equation of state. For an incompressible fluid, $\rho = \text{constant}$ can also be considered an equation of state.

Viscosity μ is commonly measured by performing the experiments Newton conceptually described when he proposed the Newtonian model: put the fluid of interest between two parallel smooth surfaces which are a small distance d apart, and slide one surface relative to the other (in the plane of the

surface) with low subsonic velocity U . Measure the shear stress $\Pi_{2,1}$ acting on the plate. The viscosity of the fluid is then given by $\mu = \Pi_{21}d/U$, after having verified that the fluid velocity profile is linear, and that the no-slip condition at the two surfaces are satisfied. To be convinced that μ is a material property, experiments with different d 's and U 's should be performed. They should all produced the same μ when the same fluid is being dealt with if the fluid is Newtonian.

For air, the kinematic viscosity $\nu = \mu/\rho$ at one atmosphere and zero degree centigrade is approximately 13.6×10^{-6} meter²/second or 146×10^{-6} feet²/second. Note that the unit of ν is velocity times length. Simply kinetic theory predicts that ν for simple gases is approximately $0.5a\ell$, where a is the averaged thermal (random) velocity of the gas molecules ($\approx \sqrt{\gamma RT}$) and ℓ is the “mean-free-path.” In fact, this formula is often used to estimate the mean-free-path of the gas under consideration.

Estimate the mean-free-path of air at one atmosphere and zero degree centigrade. Also, estimate the collisional mean-free-time.

But, as discussed earlier, the Navier-Stokes stress tensor “predicts” more than just shearing stress; when the fluid is in motion, viscous *normal* stresses are predicted by this stress tensor. This is the consequence of imposing the “common sense” requirements on Π_{ij} , in particular, the “symmetry” requirement (how about asking me (or looking it up) *why* symmetry is required?). What is the empirical evidence that the viscous normal stress predicted by the Navier-Stokes tensor is actually true?

The simplest way to test the normal stress predictions of the Navier-Stokes stress tensor is to consider a one-dimensional problem with $u(x)$ and $v = w = 0$. When u does vary with x , then Π_{11} appears (while no shearing stress at all is present). What are the consequences of having these predicted viscous normal stresses?

The short answer is: if the flow is subsonic far upstream and the Reynolds number is large, nothing much happens (if the Reynolds number is very small, then such normal stresses are the controlling terms). But if the flow is supersonic far upstream, then the possibility of a *shock wave* arises. The detailed internal structure of the shock wave can be obtained from the Navier-Stokes Equations—using viscosity measured in low subsonic shearing experiments! So: the validity of the viscous normal stresses in Π_{ij} can be verified experimentally!

1.1 Galilean Invariance

If a physical law is correct, then the equations used by a theoretician observing it while standing still and another theoretical observing it in a moving (constant velocity) frame should be related by a mathematical transformation. The concept is called Galilean Invariance.

Let x, y, z, t be the coordinate system of the first theoretician, and the fluid velocity in this frame be denoted by u, v, w . For the second theoretician, the coordinate system is x', y', z', t' in the frame moving in the $+x$ direction with velocity U . We have:

$$x = x' + Ut', \quad (1)$$

$$y = y', \quad (2)$$

$$z = z', \quad (3)$$

$$t = t'. \quad (4)$$

(Can you get $x' = x'(x, y, z, t), y' = y'(x, y, z, t) \dots$, etc. from the above?) It is very important that you introduce y', z' and t' formally so that they are not confused with their unprimed counter parts—even though they are numerically the same thing. Using the chain-rule of calculus, we have:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \quad (5)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad (6)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}, \quad (7)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - U \frac{\partial}{\partial x'}. \quad (8)$$

Note! Even though $t = t'$, the last of the above equations says their respective partial time derivatives are not the same!!!!

You can readily “derive” or be convinced that:

$$u = u' + U, \quad (9)$$

$$v = v', \quad (10)$$

$$w = w', \quad (11)$$

where u', v', w' are the velocity components of \mathbf{q}' observed by the second theoretician.

1.2 Exercises

1. Transform the continuity equation from the first frame to the second frame, and show that the equation is Galilean Invariant.
2. Transform the Navier-Stokes stress tensor, and show that it is Galilean Invariant.
3. Transform all three cartesian components of the momentum equation, and show that it is Galilean Invariant.

2 Multi-Component Fluid Mechanics

Now that you know how to write down Laws of Physics of a single homogeneous fluid (state the Laws first in normal English, pick an arbitrary glob of fluid as your system, apply the law to it, use Reynolds Transport Theorem and the Divergence Theorems to get a volume integral over the arbitrary control volume—fixed in space—which equals to zero, ...), you should be able to write down the governing equations for each component of a multi-component fluid, such as a gas mixture (*e.g.* air consists of oxygen, nitrogen, carbon dioxide, nitric oxide, etc.).

You can simply follow the same procedure as before. But there are subtle pitfalls to be noted when the components are chemically active.

We shall use α as a subscript to denote the α -th component of a fluid consisting of A number of “reacting” components. When chemical reactions are present, species can be created or removed. The continuity equation is readily derived:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{q}_\alpha) = C_\alpha - R_\alpha. \quad \alpha = 1, \dots, A, \quad (12)$$

where C_α and R_α represents the amount of α mass being created or removed per unit volume per unit time as a consequence of gas-phase reactions, respectively. You asked your chemical kinetics friends to help you with these terms. This is straightforward.

What about the momentum equation for the α -th species?

I have already talked about the issues involved in class. Below is a brief summary:

1. You must include the “collisional momentum exchange” term which represents the amount of α momentum exchanged with all non- α species. This term must be Galilean Invariant. The concept of collisional mean-free-time is involved. This collisional term exists even if the gas mixture is not chemically active. Once you realize that the α species can be created or removed, you must account for the momentum being created or removed by the action of chemical reactions. If you forgot to do that, your equations will *not* be Gallilean Invariatne.
2. The fluid velocity \mathbf{q} is now *defined* as the mass averaged velocities of all the species in the gas mixture. The fluid pressure p is now defined as the sum of the partial pressures. And the fluid enthalpy
3. In the energy equation formulation, you must decide how to deal with chemical energy released as a consequence of chemical reaction (*i.e* how to keep your books and deal with “heats of formation”).

Most importantly, I stressed in class that when one is dealing with problems with very, very small mean-free-paths (the continuum problems), the resulting “exact” governing equations are *impossible* to deal with by the most powerful computers in the world for the next centuries to come. To make progress, the intelligent thing to do is to take advantage of the smallness of mean-free-path or mean-free-time.

The bottom line is that the leading approximation to all \mathbf{q}_α 's is \mathbf{q} , the mass averaged velocity of the whole fluids. The departure from the leading order approximation can be derived from the multi-component momentum equations, yielding the classical “Fick’s Law” of diffusion—complete with formula for the diffusion coefficients.

I showed in class the “derivation” of the boundary condition on a catalytic surface for the ρ_α 's: you must provide a detailed model on what happens in the last mean-free-path so that you can equate the mass flux in this very, very narrow strait to the mass flux coming in by diffusion.

I showed how to evaluate the performance of a catalytic wall—a wall that eats every pollutant is not very much a better performer than one that eats every sixth pollutants that collided with the wall.

A wall is said to be non-catalytic with respect to the α species, when the α -th diffusion flux across the wall is “negligible” (compared to what?). So, a rusting wall is usually considered non-catalytic with respect to oxygen, even though it does absorb oxygen.