Notes #3a
MAE 533, Fluid Mechanics
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1 The First and Second Laws of Thermodynamics

Let me identify a tiny glob of fluid with mass m. The volume it occupies is denoted by V, which equals to m/ρ. Something happens to this glob of fluid thorough a reversible process so its state variables have changed.

The First Law says there exists a state variable e, called the internal energy (per unit mass) of the glob, such that:

$$mde = \delta Q - pdV \tag{1}$$

where δQ is the amount of heat added, and p is the "thermodynamic" pressure (also a state variable) of the glob. The Second Law says:

$$mds = \left(\frac{\delta Q}{T}\right)_{\text{rev}}, \tag{2}$$

where s is the entropy (per unit mass) of the fluid in question, and is guaranteed by the Second Law also to be a state variable. Putting these two equations together, we have:

$$Tds = de + pd\left(\frac{1}{\rho}\right). \tag{3}$$

Equation (3) is called the *differential equation of state* for a substance in thermodynamic equilibrium. Even though it was arrived at by thinking about a reversible process, this equation is valid also for irreversible processes!!! (this is a subtle point!).

If this glob of fluid is moving so that its state variables are time and space dependent, then a most useful form of this equation of state is:

$$T\frac{Ds}{Dt} = \frac{De}{dt} + p\frac{D}{Dt}(\frac{1}{\rho}) \tag{4}$$

2 Getting Entropy Involved

The Navier-Stokes continuity equation is:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{q} = 0. \tag{5}$$

Its momentum equation is:

$$\rho\frac{D\mathbf{q}}{Dt} = -\nabla p + \nabla\cdot\mathbf{\Pi} \tag{6}$$

and the energy equation is:

$$\rho\frac{D}{Dt}\left(e + \frac{1}{2}q^2\right) = \nabla\cdot(k\nabla T) - \nabla\cdot(p\mathbf{q}) + \nabla\cdot(\mathbf{\Pi}\cdot\mathbf{q}) \tag{7}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla \tag{8}$$

is the well-known shorthand for substantial derivative. The physical dimension of (6) is force per unit volume, while the physical dimension of (7) is energy per unit volume per unit time. As these equations stand, entropy s (or enthalpy h) is not involved.

Let us compute the inner (dot) product of $\mathbf{q}$ with (6). We have, by simple algebra,

$$\rho\frac{D}{Dt}\left(\frac{1}{2}q^2\right) = -\mathbf{q}\cdot\nabla\cdot p + (\nabla\cdot\mathbf{\Pi})\cdot\mathbf{q} \tag{9}$$

which now also has the physical dimension of energy per unit volume per unit time. Subtracting (9) from (7) and using (5), we obtain:

$$\rho\left(\frac{De}{Dt} + p\frac{D}{Dt}(\frac{1}{\rho})\right) = \nabla\cdot(k\nabla T) + \Phi \tag{10}$$

where we have done a bit of house cleaning to put all the viscous-heat conduction terms on the right hand side. The last term on the right hand side Φ is called the *viscous dissipation function*, and is given by:

$$\Phi \equiv \mathbf{\Pi} \cdot \nabla \mathbf{q} = \Pi_{ij} \frac{\partial q_i}{\partial x_j} = \frac{1}{2} \Pi_{ij} \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right) \tag{11}$$

and the summations over i and j have been omitted using Einstein's summation convention. The sign of Φ is always positive when μ is positive.

Using the differential equation of state as given by (4), we have:

$$T \frac{Ds}{Dt} = \nabla \cdot (k \nabla T) + \Phi. \tag{12}$$

The equation says what we intuitively knew all the time: if heat conduction and viscous effects are negligible *throughout the trajectory* of the tiny glob as it moves from point one to point two, then the entropy of this identified tiny glob of fluid at point two is the same as its original value at point one.

So, if you want to know whether the entropy at a downstream station of a steady flow streamtube is the same as the entropy at the upstream station, you must find out whether the heat conduction and the dissipation terms throughout the intermediate trajectory between the two points on interest are negligible.

3 Perfect Gases

Under normal temperature and pressure, most gases obeys the so-called *Ideal Gas Law*:

$$p = \rho R T \tag{13}$$

where

$$R \equiv \frac{\mathcal{R}}{\mathcal{M}} \tag{14}$$

where R is the *Gas Constant*, $\mathcal{R}$ is the *Universal Gas Constant* (8.3135 KJ/Kg-mole-K° = 1545 ft-lb/lbm-mol R° = 1.9842 BTU/lbm-mol R°),and $\mathcal{M}$ is the molecular weight of the gas (for air, $\mathcal{M} = 28.967$/mole).

When the Ideal gas law is obeyed, the internal energy e of the ideal gas is a function of temperature T only. Hence, we have $e = e(T)$—*i.e.* e does not depend on density if T is held constant. If one plots experimental data of e

versus T, the slope of the curve, denoted by C_v (usually called specific heat at constant volume), is quite constant—except that in certain temperature ranges the slope switches from a lower value to a higher value. When the temperature range of interest is limited (and not in the range of changing C_v), a good approximation is

$$e = C_v T + e_o \tag{15}$$

where C_v and e_o are constants.

If one plots experimental data of enthalpy h versus T, the resulting graph is similar. The slope of the curve is denoted by C_p (usually called specific heat at constant pressure). When the temperature range of interest is limited (and not in the range of changing C_v), a good approximation is

$$h \equiv e + \frac{p}{\rho} = C_p T + h_o \tag{16}$$

where C_p and h_o are constants. For an ideal gas, we have

$$C_p = C_v + R. \tag{17}$$

The ratio of C_p to C_v is denoted by γ:

$$\gamma \equiv \frac{C_p}{C_v}. \tag{18}$$

For monatomic gases such as argon, $\gamma = 5/3$. For diatomic gases, including air, $\gamma \approx 1.4$. In general, the dimension and order of magnitude of C_p and C_v are the same as the gas constant R.

In this course, we shall assume—unless specifically mentioned—that the gas we are dealing with obeys the ideal gas law and has constant C_v, C_p and γ.

3.1 Exercise

The speed of sound, denoted by a, can be shown to be:

$$a = \sqrt{\frac{\gamma p}{\rho}}. \tag{19}$$

Find the values of a at room temperature ($20C^\circ$) for air and for helium.