Notes #4a
MAE 533, Fluid Mechanics

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1 The One-dimensional Continuity Equation

The one-dimensional steady flow continuity equation for a perfect gas is:

$$\begin{aligned} \dot{m} &\equiv \rho A u \qquad (1) \\ &= \sqrt{\frac{\gamma}{RT^\circ}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} p^\circ A D(M;\gamma) \qquad (2) \end{aligned}$$

where

$$D(M;\gamma) \equiv \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M}{(1+\frac{\gamma-1}{2}M^2)^{\frac{\gamma+1}{2(\gamma-1)}}}. \qquad (3)$$

The values of T° (stagnation temperature) and p° (stagnation temperature) at any station are usually determined by energy (transport across the boundary) and entropy (presence of dissipations) considerations. Equation (2) can be applied between stations #1 and #2, regardless of whether the streamtube is “straight” or not. Equating the mass flow rate at the two stations, we have:

$$\frac{p_2^\circ A_2 D(M_2;\gamma)}{\sqrt{T_2^\circ}} = \frac{p_1^\circ A_1 D(M_1;\gamma)}{\sqrt{T_1^\circ}}. \qquad (4)$$

1.1 Exercises

1. There is a big tank with pressure p° and temperature T°, and there is a small nozze between the tank and the atmosphere of the room (with pressure p_a). The minimum "throat" area of the nozzle is A_*, and the nozzle area at the exit plane is A_e. The ratio A_e/A_* is 10. Find p°/p_a such that the flow is sonic at the throat and the flow is subsonic at the exit.

2. Suppose the tank pressure is precisely what you computed above. Now suppose further that somehow you are able to *increase* A_* without changing A_e. What happens to the mass flow rate? Now suppose even further that somehow you are able to *decrease* A_* without changing A_e. What happens to the mass flow rate? Assume always the exit Mach number is subsonic.

2 The One-dimensional Momentum Equatoin

We now consider a straight streamtube, *i.e.* the flow velocity vectors at stations #1 and #2 are parallel and are pointing in the $+x$ direction. We draw a control volume enclosing the streamtube between these two stations. The conservation of mass is:

$$\dot{m} = \rho_1 A_1 u_1 = \rho_2 A_2 u_2. \tag{5}$$

The Law of Conservation of Momentum, in fluid mechanics language (and limited to steady flows), is (the x-component):

> The net external force (vector) acting on a control volume equals the to net outflux of momentum (vector) across the surface of the control volume.

Putting the above in terms of mathematics, we have:

$$p_1 A_1 - p_2 A_2 + f_x = \dot{m}(u_2 - u_1) \tag{6}$$

where f_x is any "external force" acting on the control volume between the two stations—not including the the pressure forces acting on the streamtube cross-sectional surface at the two stations which have already been accounted for separately. For example, if $A_1 \neq A_2$, then f_x can represent the

x-component of the fluid pressure force acting on the walls of the streamtube ($f_x > 0$ means there is thrust!). If the streamtube wall is a solid wall, then the viscous frictional force contributes a negative term to f_x.

Equation (6) can be rewritten as, with the help of (5):

$$f_x = A_2(p_2 + \rho_2 u_2^2) - A_1(p_1 + \rho_1 u_1^2). \tag{7}$$

The entity $A(p + \rho u^2)$ is sometimes called the *stream-thrust*. It has the dimension of force. Most emphatically, (7) has *no relation whatsoever* to the incompressible Bernoulli's equation; note that there is no 1/2 factor in the second term.

We are reminded here that the Mach Number and the stagnation quantities are preferred flow variables. We rewrite $p + \rho u^2$ as (assuming perfect gas):

$$p + \rho u^2 = p^\circ G(M;\gamma) \tag{8}$$

where $G(M;\gamma)$ is defined by:

$$G(M;\gamma) \equiv \frac{1 + \gamma M^2}{\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}} \tag{9}$$

Equation (6) can now be rewritten as:

$$f_x = p_2^\circ A_2 G(M_2;\gamma) - p_1^\circ A_1 G(M_1;\gamma). \tag{10}$$

It is important to note that (10) is valid regardless of energy considerations (T° need not be constant) or entropy considerations (heat conduction and viscous dissipations may be present between the two stations).

An useful alternative representation of (10) is:

$$p_2^\circ A_2 G(M_2;\gamma) = (1 + C_f) p_1^\circ A_1 G(M_1;\gamma) \tag{11}$$

where the dimensionless C_f is defined by:

$$f_x = C_f p_1^\circ A_2 G(M_2;\gamma) \tag{12}$$

2.1 Exercises

1. Plot $G(M;\gamma)$ for $\gamma = 1.4$ over the range $0 \leq M \leq 10$. Note that $G(0;\gamma) = 0$, $G(M;\gamma)$ peaks at $M = 1$, and $G(M >> 1;\gamma) \to 0$.

2. We have a convergent divergent nozzle between stations #1 and #2, with $A_2 = A_1$. We know the flow is adiabatic (no heat or shaft work crossing the control volume boundary) and isentropic (heat conduction and viscous dissipation between the two stations are negligible), and $M_1 = 2.0$.

   - We were told that M_2 is supersonic. Find the value of M_2. In addition, what can you tell me about the minimum $A(x)$ between the two stations?
   - We were told that M_2 is subsonic. Find the value of M_2. In addition, what can you tell me about the minimum A between the two stations? What is p_2? Find the value of f_x?

   Repeat the problem for the case $M_1 = 0.2$.

3. Take a good look at (11). When do you think f_x is "negligible" from this equation?

3 Constant Area Problems

Consider now the problem with $A =$constant, so that $A_2 = A_1 = A$. Dividing (4) by (11), we have:

$$N(M_2;\gamma) = \frac{\sqrt{T_2^\circ/T_1^\circ}}{1+C_f} N(M_1;\gamma) \tag{13}$$

where

$$N(M;\gamma) \equiv \frac{D(M;\gamma)}{G(M;\gamma)}. \tag{14}$$

3.1 Exercises

1. Plot $N(M;\gamma)$ for $\gamma = 1.4$ over the range $0 \le M \le 10$. Note that $N(0;\gamma) = 0$, $N(M;\gamma)$ peaks at $M = 1$, and $N(M >> 1;\gamma) \to$ constant. Find $N(\infty;\gamma)$.

2. What happens to a flow which is subsonic at station #1 and energy is added so that $T_2^\circ > T_1^\circ$ while C_f is negligibly small? What happens when the flow is supersonic at station #1? Can you add any heat if the flow is precisely sonic at station #1?

3. Qualitatively, what happens to p_2° (relative to p_1°) when you added heat (in general)?

4. When happens if no energy has been added or removed, but C_f is some negative number? (viscous friction between fluid and solid wall of streamtube gives rise to negative C_f).

5. Between stations #1 and #2, there is a compressor from *Isentropic, Inc* (who claims that the entropy of the fluid flowing through its machines does not change). For the case $T_2^\circ = 1.2T_1^\circ$, $M_1 = 0.1$, find C_f.

Note that when you add or remove heat, no external force to the control volume need be incurred. If you add (or remove) mechanical energy by compressor, turbine or similar devices, some f_x is necessarily generated.

4 Normal Shock Relations

For the simplest case of $T_1^\circ = T_2^\circ$ and $C_f = 0$, we have:

$$N(M_2;\gamma) = N(M_1;\gamma). \tag{15}$$

This equation gives the downstream Mach number M_1 as a function of the upstream Mach number M_2 for an adiabatic flow with no external force in a constant area streamtube. No assumption on the entropy of the fluid was made. One obvious exact solution of this algebraic equation is $M_2 = M_1$—*i.e.* all flow variables at station #2 are unchanged from their values at station #1 (temperature T_2, stagnation and static pressure p_2° and p_2, and entropy s_2).

But, as can be seen from your $N(M;\gamma)$ versus M plot, there is another solution to (15) (with M_2 on the "other" side of Mach one)! This solution is:

$$M_2^2 = \frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)}. \tag{16}$$

The thermodynamic variables at station #2 can easily be computed:

$$\begin{aligned} \frac{p_2}{p_1} &= 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1), & (17) \\ \frac{\rho_2}{\rho_1} &= 1 + \frac{2(M_1^2 - 1)}{2 + (\gamma - 1)M_1^2} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2}, & (18) \end{aligned}$$

$$\frac{s_2 - s_1}{R} = \ln\left\{\left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]^{\frac{1}{(\gamma-1)}} \left[\frac{(\gamma-1)M_1^2 + 2}{(\gamma+1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\right\}. \quad (19)$$

The following are interesting observations:

- $s_2 - s_1$ is positive when $M_1 \geq 1$, and is negative when $M_1 < 1$. Hence, this solution is valid only when $M_1 \geq 1$.
- When $M_1 - 1$ is small, the above formulae can be approximated by:

  $$\frac{p_2}{p_1} \approx 1 + \frac{4\gamma}{\gamma+1}(M_1 - 1) + \ldots, \quad (20)$$

  $$\frac{\rho_2}{\rho_1} \approx 1 + \frac{4}{\gamma+1}(M_1 - 1) + \ldots, \quad (21)$$

  $$\frac{s_2 - s_1}{R} \approx \frac{16\gamma}{3(\gamma+1)^2}(M_1 - 1)^3 + \ldots. \quad (22)$$

  We see that $(s_2 - s_1)/R$ is of the order of a small number raised to the third power. Hence, when $M_1 - 1$ is small, then $(s_2 - s_1)/R$ is very small—the flow is very nearly isentropic.
- When M_1 is very large, p_2/p_1 is also very large—it is $O(M_1^2)$. However, in the large M_1 limit, ρ_2/ρ_1 is a constant (asymptotes to $(\gamma+1)/(\gamma-1)$).
- When M_1 is very, very large, what would be a good approximation to M_2?

4.1 Exercises

1. Find T_2/T_1 across a normal shock as a function of M_1.
2. Imagine a meteor crashing into our atmosphere, reaching sea level at $M = 20$. Find the static temperature (T) and pressure (p) at the stagnation point. In the moving frame of the meteor, there is a bow shock wave. You may assume that the streamtube from upstream infinity to the stagnation point passes through a normal shock. (How do you justify using the steady flow assumption in your analysis since the meteor is obviously decelerating?)

4.2 The Rankine-Hugoniet Relations

What happens when we do not have a perfect gas? It is possible to formulate the analysis of the "jump condition" across the "discontinuity" without getting the equation of state of the gas deeply involved.

Let m denote the mass flow rate per unit streamtube area. So we have from the continuity equation:

$$m = \rho_1 u_1 = \rho_2 u_2. \tag{23}$$

The momentum equation ($p+\rho u^2$=constant) can be rewritten (by eliminating u_1 and u_2 in favor of m):

$$m^2 = -\frac{p_2 - p_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}}. \tag{24}$$

The energy equation (h° =constant) can similarly be written as:

$$h_2 - h_1 = \frac{p_2 - p_1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{25}$$

Equation (25) can be stated in English as follows: the rise of static enthalpy equals to the pressure rise times the algebraic average of the specific volumes across the discontinuity.

Equations (24) and (25) are valid for any equations of state $h = h(p, \rho)$. They are called the Rankine-Hugoniet relations. When the equation of state (however complicated) is used to eliminate h from (25), the resulting algebraic relation between p_2 and ρ_2 yields a curve in the $(p, 1/\rho)$ space. This curve gives the locus of all solutions associated with the same initial condition (station #1) but different values of m or M_1. This curve is called the Hugoniet Curve.

5 The Speed of Sound

We have been talking about the speed of sound, and have been using simple formula for its computation when the gas in question is a perfect gas. How does one derive these formulae?

First of all, we recognize that sound is a pressure wave, and a very weak pressure wave at that. Atomospheric pressure is 14.7lbs/in^2 or 2116.8lbs/ft^2. The loudest sonic booms by a supersonic aircraft generates a pressure rise of about 2lbs/ft^2. The pressure amplitude of very loud music is probably 10^{-5} times the atmospheric pressure.

We consider an atmosphere of a single gas originally at rest and in thermodynamic equilibrium. We define the primed variables by:

$$
\begin{aligned}
p &= p_o + p', && (26)\\
\rho &= \rho_o + \rho', && (27)\\
\mathbf{q} &= \mathbf{q}' && (28)
\end{aligned}
$$

where subscript o denotes the undisturbed condition. We agree that, for sound waves, $p' << p_o$ and $\rho' << \rho_o$. The continuity equation can be "linearized" to become:

$$\frac{\partial \rho'}{\partial t} + \rho_o \nabla \cdot \mathbf{q}' \approx 0. \tag{29}$$

The inviscid momentum equation (*i.e.* the viscous stress term is neglected) can be "linearized" to become:

$$\rho_o \frac{\partial \mathbf{q}'}{\partial t} \approx -\nabla p'. \tag{30}$$

Neglecting also the heat conduction term, the entropy equation (derived by using both the energy and the momentum equation) can be integrated to yield:

$$s \approx s_o. \tag{31}$$

We now choose to represent the equation of state in the following form:

$$p = p(\rho, s). \tag{32}$$

Taking the divergence of (30), we have:

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla \cdot (\nabla p'). \tag{33}$$

We just need to eliminate one of the two dependent variables (for example, ρ') to obtain a single PDE for a single unknown.

Since s is a constant over all time and space, the equation of state (32) provides just the needed equation:

$$p' = p(\rho_o + \rho', s_o) - p_a. \tag{34}$$

Linearizing this equation, we have

$$p' \approx a_o^2 \rho' \tag{35}$$

where

$$a^2 \equiv \left(\frac{\partial p}{\partial \rho} \right)_s . \tag{36}$$

Note that a as defined is strictly a function of the thermodynamic variables of the fluid. We shall show presently that it is indeed the speed of sound.

Using (36) to eliminate ρ' from (33), we obtain:

$$\frac{1}{a_o^2} \frac{\partial^2 p'}{\partial t^2} = \nabla^2 p' \tag{37}$$

where we have use the *Laplacian* operator ∇^2 to represent the $\nabla \cdot \nabla$ operator.

I expect every one in this course to recognize that this is the *wave equation.* The speed of the wave is a_o.

5.1 Exercises

1. Let ϵ be a small number. Convince yourself that terms that contain second and higher power of ϵ is smaller than terms that contain only ϵ to the first power.
2. Consider the one-dimensional wave equation so that $p' = p'(x, t)$. Show that $p = f(x - a_o t) + g(x + a_o t)$ is an exact solution where $f(\xi)$ and $g(\eta)$ are twice differentiable arbitrary function of their arguments.
3. The middle A key of a piano oscillates at 440 cycles per second. Find the wave length. Assume standard conditions for the atmosphere.
4. Find $p = p(\rho, s)$ for a gas that obeys the ideal gas equation of state.