Notes #4b
MAE 533, Fluid Mechanics

S. H. Lam
email: lam@princeton.edu
Course URL: http://www.princeton.edu/∼lam

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1 Eugene Parker's Theory of Solar Wind

Assume the atmosphere of the sun to be steady and spherically symmetric. We consider one radial streamtube and denote the distance from the center of the sun by r.

The continuity equation is $\dot{m}4\pi\rho u r^2$. Taking its logarithmic derivative with respect to r, we have:

$$\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{u}\frac{du}{dr} + \frac{2}{r} = 0. \tag{1}$$

The r-momentum equation is:

$$\rho u \frac{du}{dr} = -\frac{dp}{dr} - \frac{\rho G M_o}{r^2} \tag{2}$$

where the second term on the right hand side is the gravitational term with G =the universal gravitation constant and M_o=mass of the sun. Viscous forces have been neglected (the Reynolds number is incredibly large). Instead of the energy equation, we shall make the simplifying *ad hoc* assumption (made by Eugene Parker) that the temperature T of the whole solar atmosphere is a constant (this is done to vastly simplify the algebra):

$$T = T_o. \tag{3}$$

In addition, we shall assume the perfect gas law $p = \rho RT$. Taking its logarithmic derivative with respect to r and noting that T is a constant, we have:

$$\frac{1}{p}\frac{dp}{dr} = \frac{1}{\rho}\frac{d\rho}{dr}. \tag{4}$$

Summarizing: we have three unknowns u, p and rho, and we have three ordinary differential equations (ODE), (1), (2) and (4). We are ready to go!

2 Theoretical Developments

Manipulating the ODEs, we can arrive at the following single ODE for u:

$$\frac{1}{u}\frac{du}{dr} = \frac{2a^2(r - r_*)}{r^2(u^2 - a^2)} \tag{5}$$

where

$$\begin{aligned} r_* &\equiv \frac{GM_o}{2a^2}, \quad \text{(its unit is length)}, && (6) \\ a^2 &= RT, \quad \text{(square of isothermal speed of sound, a constant here)}. && (7) \end{aligned}$$

To obtain the solution of our problem, we need only to integrate (it can be done analytically) (5)—provided we know how to find the initial condition (the value of u at $r = r_o$). To get an idea of the type of solutions to be obtained, the method of isoclines (phase portrait) is most useful.

Once $u(r)$ is found, the density $\rho(r)$ can be found by the so-called isothermal Bernoulli's equation (obtained by integrating (2)). We have:

$$\frac{\rho}{\rho_o} = \exp\left[-\frac{2r_*}{r_o}\left(1 - \frac{r_o}{r}\right) - \frac{1}{2a^2}(u^2 - u_o^2)\right]. \tag{8}$$

2.1 Static Solar Atmosphere

What about the possibility that $u = 0$ everywhere—the solar atmosphere is static? This simple solution certainly satisfies (5). The difficulty is that the resulting density $\rho(r)$ given by (8) is not credible—its prediction of the solar atmosphere at infinity is much, much too high to be acceptable.

2.2 Dynamic Solar Atmosphere

For our sun, the value of r_* is $5 - 10$ times the value of r_o, the radius of the surface of the sun. Since we know $u_o = u(r_o) << a$, the right hand side of (5) is thus positive at $r = r_o$. In other words, the solar wind velocity u increases with r near $r = r_o$. The trouble is: we don't know the value of u_o except that it must be very small in comparison to a, the isothermal speed of sound.

Looking at (5) more carefully, we see that both the numerator and the denominator of its right hand side can go to zero. If the denominator goes to zero—the flow is Mach one with respect to a—then we must require that this event occurs at $r = r_*$ because otherwise the solution is unacceptable.

So for our sun, the situation is now clear. The solution curve of (5) is the one that goes smoothly from a subsonic u near the sun's surface to "choke" at $r = r_*$, and goes on further outward supersonically beyond r_*. If the interstellar space is unhappy with the solar wind pressure at infinity, a "shock wave" can exist to adjust matters to make everybody happy. Near the sun's surface, $u << a$, and the atmosphere is quite accurately described by a hydrostatic model.

Since the earth is over 200 r_o away from the sun, Eugene Parker predicted that there is a supersonic solar wind at the orbit of the earth, and that there must be a huge bow shock wave when the solar wind meets the earth.

3 What About the Moon?

What happens if $r_o > r_*$? You can easily convince yourself that such a heavenly body cannot maintain an approximately hydrostatic atmosphere near the surface.

4 Lessons Learned

In the modern age of computers, it is more and more fashionable to numerically integrate differential equations to obtain solutions. This example shows that theoretical analysis can provide much insights that would be difficult to extract from numerical results.

In fact, how would you tackle this problem using computers?