

Notes #6

MAE 533, Fluid Mechanics

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1 Different Ways of Representing T°

The speed of sound, a , is formally defined as $\sqrt{(\partial p / \partial \rho)_s}$. It is a function of that thermodynamic state variables of the gas.

The stagnation enthalpy of a steady flow streamtube is defined by”

$$h^\circ \equiv h + \frac{1}{2}u^2. \quad (1)$$

It depends not only of the thermodynamic state of the gas, but also on the magnitude of the flow velocity u . Using the perfect gas approximation ($h = C_p T$, $C_p = \gamma R / (\gamma - 1)$, $a^2 = \gamma R T$), we have:

$$h^\circ = C_p T \left(1 + \frac{\gamma - 1}{2} M^2\right). \quad (2)$$

where $M = u/a$ is the Mach Number.

We now define a_o to be the *stagnation speed of sound*—*i.e.* the speed of sound when the gas is hypothetically brought to rest adiabatically. For a perfect gas, we have $h^\circ = C_p T_o$. Hence

$$a_o^2 = \gamma R T^\circ = a^2 \left(1 + \frac{\gamma - 1}{2} M^2\right). \quad (3)$$

In other words, a_o is a measure of the stagnation temperature of the gas.

We next define a_* to be the *critical speed of sound*—*i.e.* the speed of sound when the gas is hypothetically brought to Mach one. Setting $a = a_*$ and $M = 1$ in (3), we have:

$$a_o^2 = \gamma RT^o = \frac{\gamma + 1}{2} a_*^2. \quad (4)$$

We have previously be introduced to the concept of *ultimate velocity* u_{ult} defined by:

$$u_{\text{ult}}^2 = 2h^o. \quad (5)$$

It is the velocity that would be achieved in the steady flow stream tube when the gas temperature is zero. For a perfect gas, we have

$$u_{\text{ult}}^2 = \frac{2}{\gamma - 1} a_o^2 = \frac{\gamma + 1}{\gamma - 1} a_*^2. \quad (6)$$

Thus, a_o , a_* and u_{ult} are alternative measures of the stagnatin temperature of the gas.

A new Mach Number, M_* , defined in terms of the critical speed of sound, is useful for oblique shock discussions. We have:

$$M_* \equiv \frac{u}{a_*}. \quad (7)$$

Note that M_* is dimensionless, is supersonic or subsonic when its value is larger or less than unity (just like M). Note that when $M \rightarrow \infty$, we have $u \rightarrow u_{\text{ult}}$ and the associated value of M_* is finite!

2 Oblique Shocks

A normal shock is called a normal shock because the surface of discontinuity is normal (perpendicular) to the direction of the upstream velocity vector. When a normal shock is observed by an observer moving with velocity U_{\parallel} along the surface of the shock wave, it would appear to the moving observer as an *oblique shock*. The surface of the shock is then oblique to the unstream velocity vector, and the direction of the fluid velocity vector now changes abruptly across the shock surface. Obviously, the component of the upstream and downstream velocity vectors perpendicular to the shock surface obey the normal shock relation, the relevant Mach number being the

upstream “normal Mach Number”—the normal component of the upstream Mach Number. Equally obviously, the parallel components of the upstream and downstream velocity vectors are identical (unchanged across the shock surface).

It should be obvious that the relations across oblique shocks can be obtained from normal shock relations supplemented by some trigonometry. In all fluid mechanics text books that deal with compressible flows, one can find normal shock tables, giving the ratio of pressure, density, etc. across the shock as a function of the upstream Mach Number (for one value of γ , usually 1.4). But no text book gives tables for oblique shocks—because the number of tables needed to represent all interesting upstream Mach Number and the obliqueness is simply too large. Instead, the oblique shock relations are usually presented graphically in a diagram commonly referred to as the *shock polar*. A shock polar is a graphical representation of the downstream velocity vector for a fixed, given upstream velocity vector across a plane shock wave.

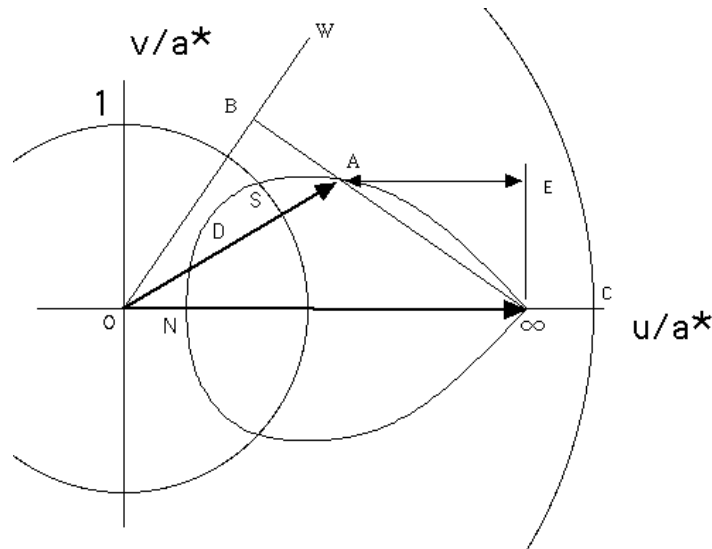


Figure 1: Oblique Shock Polar

Fig. 1 is a sketch of a shock polar. The coordinates are the velocity components in the x and y directions, normalized by a_* . The unit circle therefore separates subsonic flows (inside the unit circle) from supersonic flows downstream of the oblique shock. The outer big circle with radius OC has length u_{ult}/a_* , and represents the maximum possible value of flow velocity from the

point of view of the energy equation (on this big circle, the fluid temperature is zero). The upstream velocity vector, non-dimensionalized by a_* , is represented by the line $O-\infty$. The locus of all possible oblique shock solutions for the downstream velocity vector is given by the tear-drop shape figure—the shock polar—with the pointed vertex of the tear-drop at the tip of the $O-\infty$ vector.

The point N represents the normal shock, the point D represents the “detachment” solution, the point S represents the “sonic” solution, and the point ∞ represents the “nothing happened” solution. Also shown on the diagram is a sample solution point A (which happens to be supersonic as shown).

In general, we have:

$$u_{\parallel 1} = u_{\parallel 2}. \quad (8)$$

The continuity equation across the shock wave says:

$$\frac{\rho_2}{\rho_1} = \frac{u_{\perp 1}}{u_{\perp 2}}. \quad (9)$$

The density ratio ρ_2/ρ_1 across the oblique shock is a function of $M_{\perp\infty}$ (for a given equation of state of the gas of interest) only and can be found from the normal shock relations.

To construct the oblique shock polar, one begins with a Cartesian coordinate system on a piece of graph paper, and locate the point ∞ to represent the upstream Mach Number M_∞ of interest. This is the also the solution point corresponding to the case of “nothing happened.”

Let the angle between the unstream velocity vector and the shock surface be denoted by σ . Hence, the point N corresponds to the normal shock solution with $\sigma = \pi/2$. For the general case (with arbitrary σ), we first draw the line OW on the diagram, the angle $\angle(W-O-\infty)$ is precisely σ . The relevant normal Mach Number is simply $M_{\perp\infty} = M_\infty \sin \sigma$, and is represented on the diagram by the line segment $B-\infty$ which is perpendicular to the line $O-W$. One can easily find the density ratio ρ_2/ρ_1 for a normal shock with this $M_{\perp\infty}$, and the value of $u_{\perp 2}/u_{\perp 1}$ is then readily obtained by the use of (9). The point A on the shock polar is obtained by $(B-A)/(B-\infty) = u_{\perp 2}/u_{\perp 1} = \rho_1/\rho_2$. The angle $\angle(A-O-\infty)$ is the “turning angle” caused by the oblique shock, and is often denoted by δ .

By changing the value of the shock wave angle σ from $\pi/2$ downward, the locus of the solution points is the tear-drop shape oblique shock polar.

2.1 The Analytical Oblique Shock Polar

The analytical formula for the oblique shock polar for a perfect gas can be found in all the text books.

$$\left(\frac{v}{a_*}\right)^2 = \left(\frac{U_\infty}{a_*} - \frac{u}{a_*}\right)^2 \frac{\frac{uU_\infty}{a_*^2} - 1}{\frac{2}{\gamma+1} \left(\frac{U_\infty}{a_*}\right)^2 - \frac{uU_\infty}{a_*^2} + 1}. \quad (10)$$

2.2 Exercises

1. Show that for given M_∞ , the smallest value of σ (corresponds to the weakest possible shock wave) is $\sigma = \arcsin\left(\frac{1}{M_\infty}\right) = \arctan\left(\frac{1}{\sqrt{M_\infty^2 - 1}}\right)$. This shock angle valid in the limit of very weak shock is called the Mach Angle, frequently denoted by β .
2. Show that it is possible to obtain the value of $p_2 - p_1$ across an oblique shock from the shock polar by measuring the value of Δu defined by:

$$\Delta u \equiv u_2 - u_1 \quad (11)$$

which is shown on Fig. 1 as the line segment A-E. Hint: use x -momentum balance as done in class.

3. When M_∞ is asymptotically large, then $M_{\perp\infty}$ is also asymptotically large except when the turning angle δ is asymptotically small. Hence, ρ_2/ρ_1 is approximately a constant $(\gamma + 1)/(\gamma - 1)$. Convince yourself that in the large M_∞ limit the shock polar is a circle (except when $\delta \ll 1$).

3 Weak Oblique Shocks

In the general case, the exact equations for the relations across oblique shocks are quite complicated. When δ is small, however, simple approximate equations are available.

First of all, the shock is asymptotically weak in this limit, and is often called a ‘‘Mach Wave’’ instead. The normal Mach Number $M_{\perp\infty}$ is approximately unity, and σ is approximately β . Using only trigonometry, we can

show that:

$$\tan \beta \approx \frac{-\Delta u}{U_\infty \delta}. \quad (12)$$

Using the formula for β given previously, we have:

$$\Delta u \approx -\frac{U_\infty \delta}{\sqrt{M_\infty^2 - 1}}. \quad (13)$$

3.1 Exercises

1. For weak oblique shock with $\delta \ll 1$, show that Δs ($(s_2 - s_1)/R$) is $O(\delta^3)$, and is therefore negligible. In other words, in the leading approximation, weak shocks are isentropic.

2. Show that

$$\frac{\Delta p}{p_\infty} \approx \frac{\gamma M_\infty^2 \delta}{\sqrt{M_\infty^2 - 1}}. \quad (14)$$

where $\Delta p \equiv p_2 - p_1$.

3. Take advantage of the fact that the flow is approximately isentropic, find $\Delta T/T$ where $\Delta T \equiv T_2 - T_1$.
4. What happens when δ is small but negative?