Notes #7
MAE 533, Fluid Mechanics

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> Just a reminder: there is no class on Friday, October 30th. The mid-term is due right after the Fall Break. You are welcome to ask me questions by email, even during the Fall Break. I will send copy of my response, minus the name of the person sending in the question, to the whole class.

1 Weak Oblique Shocks

We define the Mach Angle β by:

$$\tan\beta \equiv \frac{1}{\sqrt{M^2-1}} \quad \text{or} \quad \sin\beta \equiv \frac{1}{M}, \tag{1}$$

where M is the local Mach Number, assumed to be supersonic. We denote the local flow velocity (magnitude) by q, and its direction by δ. When a very, very weak oblique shock wave occurs, the flow direction is changed by $d\delta$ (defined positive in the *counterclockwise direction* when the vector $\mathbf{q}$ flows generally from left to right), and the flow velocity magnitude is changed by dq. Using simple geometry and trigonometry, and noting that the normal Mach Number $M_\perp \approx 1$, we have:

$$\frac{dq}{q} \approx \mp \frac{d\delta}{\sqrt{M^2-1}} \tag{2}$$

where the minus sign is to be used when the angle between the flow direction and the oblique shock is $+\beta$, and the plus sign is to be used when the angle is $-\beta$. We now assume the flow field has constant stagnation temperature (or stagnation speed of sound) everywhere. We have:

$$a_o^2 = a^2(1 + \frac{\gamma - 1}{2} M^2). \tag{3}$$

Taking its logarithmic derivative, we have:

$$\frac{da}{a} = -\frac{(\gamma - 1) M dM}{2 + (\gamma - 1) M^2}. \tag{4}$$

Since $M = q/a$, its logarithmic derivative gives:

$$\frac{dM}{M} = \frac{dq}{q} - \frac{da}{a}. \tag{5}$$

Eliminating da and dq between (2),(4) and (5), we have:

$$dM \approx \mp \frac{M \left[1 + \frac{\gamma - 1}{2} M^2\right] d\delta}{\sqrt{M^2 - 1}}. \tag{6}$$

If M and $1 >> d\delta \geq 0$ are known, (6) provides a simple formula for dM (in response to $d\delta$), which can be used in (4) and one to find the values of da and dq, respectively.

1.1 Exercises

1. Derive the leading approximation for ds, the change of entropy across a weak shock, as a function of $d\delta$.

2. Convince yourself when to use the plus sign and when to use the minus sign in (2) and (6).

2 Prandtl-Meyer Expansion Fan

All the equations in the previous section assume that $d\delta > 0$ so that the normal Mach Number is greater than unity.

What happens when $d\delta$ is asymptotically small (smaller than the smallest number you just thought of), but is negative? In the limit of $M_\perp \to 1$, (6) can be rewritten as:

$$d\theta = \pm \frac{\sqrt{M^2-1}dM}{M\left[1+\frac{\gamma-1}{2}M^2\right]}. \tag{7}$$

where $d\theta \equiv -d\delta \geq 0$. This formula does not violate the Second Law of Thermodynamics, because in this limit the asymptotically weak "expansion shock" does not decrease the fluid entropy—the flow is isentropic!

Equation (7) can be integrated analytically. We have:

$$\theta = \pm\omega(M) + \text{constant}, \tag{8}$$

where

$$\omega(M) \equiv \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan\left[\sqrt{\frac{\gamma+1}{\gamma-1}(M^2-1)}\right] - \arctan\sqrt{M^2-1}. \tag{9}$$

Note that, strictly speaking, both θ and ω are in units of radians (which is dimensionless). However, in all text books, this function is presented in tables expressed in units of degrees.

2.1 Exercises

1. Find the value of $\omega(\infty)$ as a function of γ.
2. Use whatever software you are most familiar, and contruct a table of θ versus M in unit of degrees. Use $\gamma = 1.4$
3. You have a uniform horizontal flow with undisturbed Mach number $M_1 = 2.0$. There is a flat plate with angle of attack $\alpha = 0.1$ radian. Find the pressure and temperature on the lee-side of the flat plate. (assume the oblique shock on the other side is attached).

3 The Method of Characteristics

The study of oblique shocks and Prandtl-Meyer Fan allow us to analyze quantatively the flow fields of supersonic aerodynamic flows. If you should feel that the methodology presented so far is somewhat *ad hoc*, you are certainly justified. The question is: can we do things in a more systematic fashion?

The answer is yes. For steady two-dimensional supersonic flows, there is indeed a systematic way of doing things.

What we are trying to do is to solve the governing partial differential equations which are mathematical statements of the Laws of Physics. We can write down the governing PDEs for two-dimensional compressible steady flows as follows:

$$0\frac{\partial p}{\partial x} + 0\frac{\partial p}{\partial y} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial u}{\partial x} + 0\frac{\partial u}{\partial y} + 0\frac{\partial v}{\partial x} + \rho\frac{\partial v}{\partial y} \quad = \quad 0, \tag{10a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} + 0\frac{\partial p}{\partial y} + 0\frac{\partial \rho}{\partial x} + 0\frac{\partial \rho}{\partial y} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + 0\frac{\partial v}{\partial x} + 0\frac{\partial v}{\partial y} \quad = \quad 0, \tag{10b}$$

$$0\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial y} + 0\frac{\partial \rho}{\partial x} + 0\frac{\partial \rho}{\partial y} + 0\frac{\partial u}{\partial x} + 0\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \quad = \quad 0, \tag{10c}$$

$$\frac{u}{a^2}\frac{\partial p}{\partial x} + \frac{v}{a^2}\frac{\partial p}{\partial y} - u\frac{\partial \rho}{\partial x} - v\frac{\partial \rho}{\partial y} + 0\frac{\partial u}{\partial x} + 0\frac{\partial u}{\partial y} + 0\frac{\partial v}{\partial x} + 0\frac{\partial v}{\partial y} \quad = \quad 0, \tag{10d}$$

The first equation is the continuity equation, the second and third are the (inviscid) momentum equation, and the last is $Ds/Dt = 0$ in disquise (no heat conduction, no viscous dissipation), since a^2 is defined by:

$$a^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s . \tag{11}$$

Let us now draw a line on the flow field passing through a point of interest. Consider now an adjacent point, denote the change of any variable between these two point (both on the line) using the differential operator. Thus, the change of the x-coordinate is dx, the change of the y-coordinate is dy, the change of pressure is dp, etc. Assuming that all dependent variables are differentiable, we have:

$$dx\frac{\partial p}{\partial x} + dy\frac{\partial p}{\partial y} + 0\frac{\partial \rho}{\partial x} + 0\frac{\partial \rho}{\partial y} + 0\frac{\partial u}{\partial x} + 0\frac{\partial u}{\partial y} + 0\frac{\partial v}{\partial x} + 0\frac{\partial v}{\partial y} \quad = \quad dp, \tag{12a}$$

$$0\frac{\partial p}{\partial x} + 0\frac{\partial p}{\partial y} + dx\frac{\partial \rho}{\partial x} + dy\frac{\partial \rho}{\partial y} + 0\frac{\partial u}{\partial x} + 0\frac{\partial u}{\partial y} + 0\frac{\partial v}{\partial x} + 0\frac{\partial v}{\partial y} \quad = \quad d\rho, \tag{12b}$$

$$0\frac{\partial p}{\partial x} + 0\frac{\partial p}{\partial y} + 0\frac{\partial \rho}{\partial x} + 0\frac{\partial \rho}{\partial y} + dx\frac{\partial u}{\partial x} + dy\frac{\partial u}{\partial y} + 0\frac{\partial v}{\partial x} + 0\frac{\partial v}{\partial y} \quad = \quad du, \tag{12c}$$

$$0\frac{\partial p}{\partial x} + 0\frac{\partial p}{\partial y} + 0\frac{\partial \rho}{\partial x} + 0\frac{\partial \rho}{\partial y} + 0\frac{\partial u}{\partial x} + 0\frac{\partial u}{\partial y} + dx\frac{\partial v}{\partial x} + dy\frac{\partial v}{\partial y} \quad = \quad dv, \tag{12d}$$

Now, it is clear that (10a,b,c,d) and (12a,b,c,d) together gives us eight (linear) equations for the eight partial derivatives. An elegant way of writing these equations is:

$$\mathbf{A}\mathbf{z} = \mathbf{d} \tag{13}$$

where $\mathbf{A}$ is the 8×8 matrix (excuse me for not writing it out for you), and $\mathbf{z} = [\partial p/\partial x, \partial p/\partial y, \partial \rho/\partial x, \partial \rho/\partial y, \partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y]^T$, and $\mathbf{d} = [0, 0, 0, 0, dp, d\rho, du, dv]^T$. Using the good old Kramer's rule, we have:

$$\frac{\partial p}{\partial x} = \frac{N_1}{D}, \qquad \ldots, \tag{14}$$

where D is the determinant of $\mathbf{A}$, and N_n is the determinant of $\mathbf{A}$ with the n-th column replaced by the column vector $\mathbf{d}$.

We now ask: is there the possibility that the line is drawn in such a way that $D = 0$ on this line? If so, then along this line (on which $D = 0$), we can conclude that $N_n = 0$ also.

Lines on which $D = 0$ are called *characteristics* or *characteristic lines.* In other words, dy and dx are related in a special way along characteristic lines. The relations (between dp, $d\rho$ etc.) given by $N_n = 0$ are called *characteristic relations.*

It is simply a matter of algebra to show that $D = 0$ yields a fourth-order algebraic equation for dy/dx. Hence, four "roots" are expected, yielding four characteristic lines. Two of the roots are identical, and are:

$$\frac{dy}{dx} = \frac{v}{u}. \tag{15}$$

Lines which satisfy (15) are *streamlines*—they are the trajectories followed by any tiny glob of fluid.

The remaining two roots are real only if the local Mach Number is supersonic—otherwise they are purely imaginary. In other words, if M is subsonic, $D = 0$ is not a possibility. Consequently, the method of characteristics does not apply to subsonic flows.

When the flow is supersonic, the remaining two roots are:

$$\frac{dy}{dx} = \tan(\theta \pm \beta) \tag{16}$$

where θ is the direction of the velocity vector, measured positive in the counterclockwise direction, and β is the Mach Angle.[1]

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[1] When you work out the determinant of $D = 0$, these two roots comes out in a very

The characteristic relations are ODEs which are "coupled" in a special way. In the general case, they are ODEs. Hence any characteristic line which "enters" the domain of interest must intersect an "initial value line" so that the needed "initial condition" is available. Under certain conditions, these ODEs can be integrated in closed-form to yield algebraic relations.

It is relatively easy to be convinced that the construction of solutions to a supersonic steady flow problem is a "knitting" procedure—in order to find the solution at a point near an "initial value" line, one needs to find the four (actually, three) characteristics issuing from the point of interest, see where they intersect the initial value line, and apply the appropriate characteristic relations on each of the four (actually, three) characteristic lines .... four "stitches" per adjacent point ....

Initial value lines are not allowed to coinside with characteristic lines. The reason should be completely obvious.

3.1 Exercises

1. Can you guess what the two characteristic relations are on the *streamline characteristics*? Hint: they related dp and $d\rho$, and dp and du and dv.

2. Can you guess what the two characteristic relations are on the *Mach line characteristics*?

3. For steady supersonic flows, "zone of influence" and "zone of silence" are technical terms. What do you think they mean?

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messy form. The form presented above is mathematically identical to that messy answer, but is crystal clear physically.