

Notes #7a

MAE 533, Fluid Mechanics

S. H. Lam
lam@princeton.edu
<http://www.princeton.edu/~lam>

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Repeating the reminder: there is no class on Friday, October 30th. The mid-term is due right after the Fall Break. You are welcome to ask me questions by email, even during the Fall Break. I will send copy of my response, minus the name of the person sending in the question, to the whole class.

1 Method of Characteristics (continued)

In an inviscid, steady supersonic two-dimensional flow, there are four characteristics: two of them are represented by the streamline, and the other two are represented by the Mach lines.

1.1 The Streamline Characteristics

Along a streamline, defined by:

$$\frac{dy}{dx} = \frac{v}{u} = \tan \theta, \quad (1)$$

where θ is the flow direction (positive clockwise), there are two characteristics relations:

$$dp = a^2 d\rho, \quad (2a)$$

$$dh^o = 0, \quad (2b)$$

where a is the speed of sound and h^o is the stagnation enthalpy:

$$h^o \equiv h + \frac{1}{2}q^2, \quad q^2 \equiv u^2 + v^2. \quad (3)$$

1.2 The Mach Line Characteristics

The two Mach lines are defined by:

$$\frac{d_{\pm}y}{d_{\pm}x} = \tan(\theta \pm \beta) \quad (4)$$

where β is the Mach Angle:

$$\tan \beta = \frac{1}{\sqrt{M^2 - 1}}, \quad M \equiv \frac{q}{a}. \quad (5)$$

If you stand on the streamline and face downstream, the Mach line issuing from you forward toward the right is called a right-running wave, and the other one is called a left-running wave. With this terminology, the plus and minus signs in (4) are associated with left and right running waves, respectively. On these Mach line characteristics, the characteristic relations are:

$$\frac{dq}{q} = \pm \frac{d\theta}{\sqrt{M^2 - 1}}. \quad (6)$$

Note the signs!

- Along a left-running Mach line (using the plus sign in (4)), the plus sign is used in (6).
- Along a right-running Mach line (using the minus sign in (4)), the minus sign is used in (6);

1.3 The Iso-energetic Case

When h^o is a constant over the whole flow field, the flow is said to be *iso-energetic*. For this special case, dq/q in (6) can be eliminated in favor of dM/M (using the logarithmic differential of h^o to help out, as was done previously) to yield (6) of Notes #7 (note $d\delta = -d\theta$). In other words, for

iso-energetic supersonic flows, the characteristic relations along the Mach lines can be integrated to yield:

$$\theta = \pm\omega(M) + \text{constant.} \quad (7)$$

where $\omega(M)$ is the Prandtl-Meyer function.

Note the signs!

- Along a left-running Mach line (using the plus sign in (4)), the plus sign is used in (7).
- Along a right-running Mach line (using the minus sign in (4)), the minus sign is used in (7);

2 Example

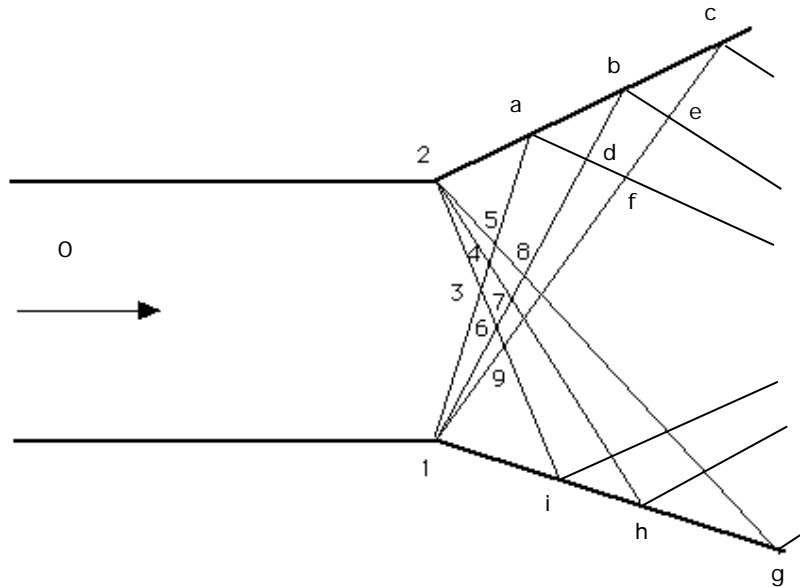


Figure 1: Steady Supersonic Flow in Channel

Consider the two-dimensional steady supersonic flow in a channel as depicted in Fig. 1. The flow in the constant area section (represented by subscript 'o'), is uniform: uniform stagnation temperature, uniform entropy, and uniform velocity (with Mach Number $M_o = 1.5$) in the $+x$ direction. In the divergent section, the turn angle of the upper wall is 0.2 radians, and the turn angle of the lower wall is -0.15 radians

The streamline characteristics fill the whole flow field. Hence we can conclude that stagnation enthalpy h^o and entropy s are constants over the whole flow field. Hence, (7) are the (integrated) characteristic relations on the Mach line characteristics.

We can roughly sketch out the “interesting” Mach lines in this flow field. The “leading” and “trailing” edges of the Prandtl-Meyer Fans can be easily drawn, and the values of the integration constants (see (7)) on these Mach lines can readily be evaluated. We can pick some intermediate values for the integration constants and sketch their associated Mach lines. In Fig. 1 we have shown only one such intermediate Mach line per Prandtl-Meyer Fan. As these Mach lines “strike” the wall, the “reflected” Mach lines are drawn. The integration constants for the reflected Mach lines can easily be determined.

Once we know the integration constants of all the Mach lines shown in Fig. 1, the values of θ and M at the intersections of Mach lines can be computed.

2.1 Exercises

1. What happens if the lower wall has a turning angle of $+0.02$ radians? Where is the oblique shock wave, and what happens to the left-running Mach lines?
2. When a flow is either precisely or approximately isentropic, the value of the integration constant on a Mach line characteristic is an invariant of that characteristic. What happens when there is a finite strength oblique shock wave present? The Mach line characteristic that crosses the shock wave will have a “kink” at the shock surface, and the integration constant on this characteristic will “jump” across the shock surface. Can you figure out how to compute the kink and the jump (just outline the strategy)?
3. What happens if either the stagnation enthalpy h^o or the entropy s is not uniform in the flow field of interest (this is a tough one)?