

Notes #8
MAE 533, Fluid Mechanics

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1 One-dimensional Unsteady Inviscid Flows

We consider now the problem of unsteady flow in one-dimension under the inviscid assumption. We have a rigid tube with cross-sectional area $A(x)$, and we assume that all the variables (x -component of velocity u and all the thermodynamic state variables) are functions of x and t only.

The governing equations are:

$$A \frac{D\rho}{Dt} + \rho \frac{\partial(uA)}{\partial x} = 0, \quad (1a)$$

$$\rho \frac{Du}{Dt} + \frac{\partial p}{\partial x} = 0, \quad (1b)$$

$$\frac{Ds}{Dt} = 0, \quad (\text{except across shocks}) \quad (1c)$$

Under the one-dimensional assumption, the substantial derivative is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}. \quad (2)$$

We assume the fluid is in thermodynamic equilibrium in the sense that an equation of state exists of each glob of fluid (*e.g.* given two state variables, any other state variables can be computed).

1.1 Exercises

1. Show that the continuity equation under the one-dimensional assumption is:

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho u A)}{\partial x} = 0. \quad (3)$$

Convince yourself that this equation is valid even if A depends on time (Hint: use a “thin” control volume consisting of a thin truncated cone with “height” dx and cross-sectional area $A(x, t)$).

2. Derive (1a) under the assumption that $A(x)$ does not depend on time.
3. Convince yourself that, unlike (1a), equation (1b) is indeed independent of $A(x)$.

1.2 The Method of Characteristics

We have a set of first order (quasi-linear) partial differential equations.¹ We can readily use the methodology we learn in two-dimensional supersonic steady flows to analysis this class of problem. In class, I showed you an algebraically simpler way of getting the characteristics and the characteristic relations for this problem (than using the method using Kramer’s rule).

There will be three characteristics: the particle path, the left-running, and the right-running. The latter two are “wave-like” characteristics. Remember: a wave is left or right running as observed by an observer moving with the fluid particle. For example, if $u > a$, a left-running wave is actually running towards the right in the laboratory frame.

With the help of the equation of state, we have just three variables in this class of problem: the kinematic dependent variable u , and two thermodynamic variables (*e.g.* a and s). Your domain of interest in the x - t plane is usually “open” in the future direction. Usually, you are supposed to know all *three* dependent variables initially ($t = 0$). Note: on the line $t = 0$, there are *three* characteristics *ENTERING* the domain of interest. Usually, the physical problem may also impose *boundary values* on space-like boundaries, such as $x = 0$ and $x = L$ (or a moving boundary such as $x = x_p(t)$). Question: how many boundary conditions are needed or allowed on such space like

¹A differential equation is quasi-linear if it dependence on the highest derivative is linear. Its dependence on the next highest derivative or on the dependent variables may be nonlinear.

boundaries? Answer: the number of boundary conditions needed or allowed equals to the number of characteristics ENTERING the region of interest. You should be able to convince yourself that the above statement is true.

If you have a space-like boundary on the right side of the domain of interest, with $u > a$, then all three characteristics are EXITING the domain of interest. In this case, NO boundary condition is needed or allowed. Of course, we had known about this intuitively a long time ago.

1.3 Exercises

We assume $A(x; t)$ is known and is given to you.

1. Derive the equation for the (3) characteristics for the general $A(x; t)$ case.
2. Derive the (3) characteristic relations for the general $A(x; t)$ case.
3. Convince yourself that when A is not a simple constant (independent of x and t), the left and right-running characteristic relations remain differential equations—there are no *Riemann Invariants* even if entropy s is a constant over the whole domain of interest because these ODEs cannot be integrated analytically.

2 The Simple $A=\text{constant}$ Case

As was shown in class, the constant area results are the following:

$$\delta_o s = 0 \quad \text{on} \quad \frac{dx}{dt} = u, \quad (4)$$

$$\delta_{\pm} p \pm \rho a \delta_{\pm} u = 0 \quad \text{on} \quad \frac{dx}{dt} = u \pm a, \quad (5)$$

where a is the speed of sound (which is a state variable). It should be obvious that δ_o means a differential change along a fluid particle path, δ_+ means a differential change along the right-running characteristic, etc. The product ρa is called *acoustic impedance*.

Under the additional assumption that entropy is a constant initially (therefore it remains a constant for $t \geq 0$ provided that no shock waves

have occurred), the characteristic relations on the wave-like characteristics can be integrated to yield Riemann Invariants. For perfect gas, we have:

$$\frac{2a}{\gamma - 1} \pm u = C_{\pm} \quad \text{on} \quad \frac{dx}{dt} = u \pm a. \quad (6)$$

Hence, for the constant entropy case, we can use C_+ and C_- as our primary unknowns. Once C_+ and C_- has been found at any intersection point of the two wave-like characteristics, the local values of u and a can readily be solved for:

$$u = \frac{1}{2}(C_+ - C_-), \quad (7)$$

$$a = \frac{\gamma - 1}{4}(C_+ + C_-). \quad (8)$$

After u and a have been computed at sufficiently large number of intersection points, the wave-like characteristics can be drawn, locating the intersection points in the physical-time domain.

What happens when the wave-like characteristics of the same family cross each other? A shock wave will emerge, dividing the region into two parts so that in each region a single pair of wave-like characteristics are in control. The value of the Riemann-Invariants *jumps* across the shock wave. If the shock wave is strong and is not perfectly straight on the $x-t$ diagram, the entropy field behind the shock will no longer be uniform, and the Riemann-Invariants there will no longer be invariants. The characteristic differential equations must then be numerically computed, one small “stitch” at a time.

If the expected shock wave is expected to be “weak,” then the “overlap” region would be tiny. Moreover, the entropy rise would be small. In such case, we can continue to use the Riemann-Invariants to obtain good approximate answers.

2.1 Acoustic Impedance $Z \equiv \rho a$

Consider a one-dimensional constant area pipe filled with two distinct gases on the two sides of the dotted line in Fig. 1. Initially, the pressure p_1 and p_2 are equal, and the flow field is quiescent. The dotted line separates the two gases. If the same gas occupies both sides, then the dotted line may represent an entropy (or temperature) discontinuity.

A weak right-running wave comes from the left side (region #1) such that $u_3 \neq 0$ but is small. This wave is transmitted to region #2 (as shown, $a_1 < a_2$ is assumed) as a right-running wave, and is reflected back toward the left in region #1 as a left-running wave. The conditions in regions #3, #4 and #5 are constant and are distinct from each other. Question: how to find the solution in regions #4 and #5?

Taking advantage of the assumption that the disturbance is weak, we use the following approximate characteristic relations:

$$d_{\pm}p \pm Z d_{\pm}u = 0 \quad \text{along} \quad \frac{dx}{dt} = u \pm a. \quad (9)$$

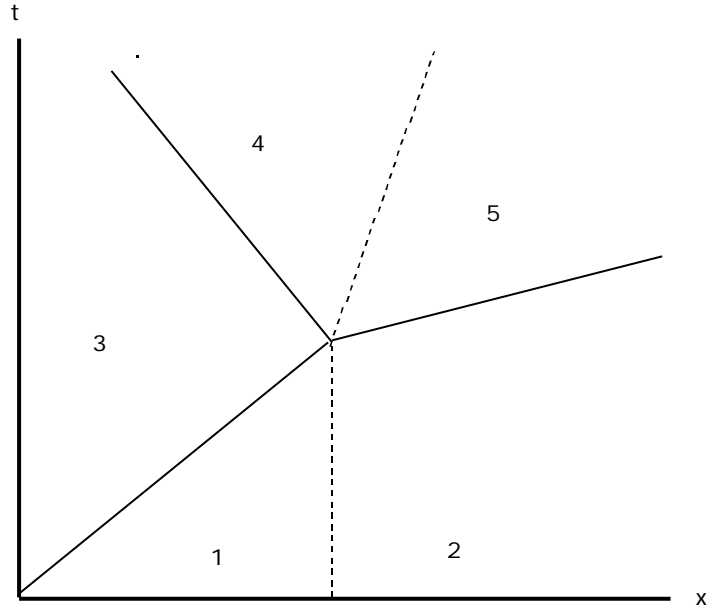


Figure 1: Wave Reflection at Fluid Interface

- Using the left-running characteristic relation between regions #1 and #3:

$$p_3 - p_1 - Z_1 u_3 \approx 0. \quad (10)$$

- Using the right-running characteristic relation between regions #3 and #4:

$$p_4 - p_3 + Z_1(u_4 - u_3) \approx 0. \quad (11)$$

- Using the left-running characteristic relation between regions #1 and #3:

$$p_5 - p_2 - Z_2(u_5) \approx 0. \quad (12)$$

We have make the self-consistent approximations $Z_1 \approx Z_3 \approx Z_4$ and $Z_2 \approx Z_5$. We require the fluid velocity and pressure to be continuous across the interface. Hence:

$$p_4 = p_5, \quad (13a)$$

$$u_4 = u_5. \quad (13b)$$

Solving these three equations, we obtain:

$$(\Delta p)_{3-1} \equiv p_3 - p_1 = Z_1 u_3, \quad (14a)$$

$$(\Delta p)_{4-1} \equiv p_4 - p_1 = \frac{2Z_1 Z_2}{Z_2 + Z_1} u_3, \quad (14b)$$

$$u_{\text{interface}} \equiv u_4 = u_5 = \frac{2Z_1}{Z_2 + Z_1} u_3. \quad (14c)$$

Subtracting (14a) from (14b) and using (14a) to eliminate u_3 , we obtain:

$$(\Delta p)_{4-3} \equiv p_4 - p_3 = \frac{Z_2 - Z_1}{Z_2 + Z_1} (\Delta p)_{3-1}. \quad (15)$$

Equation (15) is an interesting relation to remember: the pressure change across the reflected wave $(\Delta p)_{4-3}$ is proportional to the pressure change across the incident wave $(\Delta p)_{3-1}$, and the dimensionless proportional factor is a function of Z_1 and Z_2 . *If $Z_2 > Z_1$, i.e. region #2 has higher acoustic impedance, then the reflected wave pressure change has the same sign as the incident wave . . .*

What happens when the incident wave is not weak? The interface conditions of sameness of velocity and pressure still must be respected. The calculations will be slightly more involved, but the qualitative influence of the relative values of acoustic imedance will remain generally valid.