Notes #8a
MAE 533, Fluid Mechanics

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1 Acoustics in Exponential Horn

We now consider the problem of acoustics in a tube with area $A(x)$ which is not a constant. We assume initially the entropy field is uniform. Consequently, the entropy field will remain a constant since heat conduction and viscous effects are neglected. We introduce primed entities as follows:

$$u = u', \tag{1}$$
$$\rho = \rho_o + \rho', \tag{2}$$
$$p = = p_o + p' \tag{3}$$

where ρ_o and p_o are the original undisturbed values. For acoustics problems, the disturbances are, by definition, very small. Hence, the primed entities are "small," allowing us to "linearize" the equations for them. The continuity and momentum equations are (see eq.(1a,b) of Notes #8a):

$$\frac{1}{\rho_o}\frac{\partial \rho'}{\partial t} + \frac{\partial u'}{\partial x} + \frac{u'}{A}\frac{dA}{dx} \approx 0, \tag{4}$$
$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_o}\frac{\partial p'}{\partial x} \approx 0. \tag{5}$$

The fact that entropy is constant allows us to write:

$$p' \approx a_o^2 \rho'. \tag{6}$$

We now have three equations for the three unknowns p', ρ' and u'. Eliminating p' and ρ', we obtain a single partial differential equation for u' as follows:

$$\frac{1}{a_o^2}\frac{\partial^2 u'}{\partial t^2} = \frac{\partial^2 u'}{\partial x^2} + \frac{\partial}{\partial x}\left(u'\frac{d\ln A}{dx}\right) \tag{7}$$

You can readily verify that when A is a constant, the equation reduces to the general 3-D acoustic PDE derived in a previous Notes.

We now consider the special case of an exponential horn:

$$A(x) = A_o \exp(\mu x) \tag{8}$$

where μ is a positive constant with physical unit of reciprocal length, and A_o is a reference cross-sectional area. For the exponential horn, the governing PDE is then a simple constant coefficient one:

$$\frac{1}{a_o^2}\frac{\partial^2 u'}{\partial t^2} = \frac{\partial^2 u'}{\partial x^2} + \mu\frac{\partial u'}{\partial x}. \tag{9}$$

We now consider periodic acoustic waves of frequency ω (cycles/second) in this tube. We consider the trial solution:

$$u' = X(x)\exp(2\pi i\omega t) \tag{10}$$

where i is $\sqrt{-1}$. We derive the ODE for $X(x)$ by substituting (10) into (9) to obtain:

$$X_{xx} + \mu X_x + \frac{1}{\lambda^2}X = 0 \tag{11}$$

where λ is defined by:

$$\lambda \equiv \frac{a_o}{2\pi\omega} \tag{12}$$

and can be interpreted to be the *wave length* of the acoustic wave. The general solution to the second order ODE (11) is:

$$X(x) = C_+X_+(x) + C_-X_-(x) \tag{13}$$

where $C_\pm$ are (complex) constants (independent of x) and $X_\pm$ is given by:

$$X_\pm(x) = \frac{1}{\sqrt{A}}\exp(\pm\frac{i\sigma x}{\lambda}) \tag{14}$$

where σ is defined by:

$$\sigma \equiv \sqrt{1 - \frac{\lambda^2 \mu^2}{4}} \tag{15}$$

Note that σ is NOT guaranteed to be real!

Substituting (13) and (14) into (10), we obtain:

$$u' = \frac{C_+}{\sqrt{A}} \exp\left(2\pi i\omega(t + \frac{\sigma x}{a_o})\right) + \frac{C_-}{\sqrt{A}} \exp\left(2\pi i\omega(t - \frac{\sigma x}{a_o})\right). \tag{16}$$

The first term can be interpreted (more or less ??) as a LEFT-running wave and the second term a RIGHT-running wave (??). The "apparent speed" of these waves appears to be a_o/σ. These interpretations are highly dubious!

The impacts of $A(x)$ not being a constant are represented by the dimensionless parameter σ. When the wave-length λ of the acoustic wave is short in comparison to $1/\mu$, the length needed for the exponential horn to "e-fold" its cross-sectional area, σ is real and is near unity. When $\lambda\mu = 2$, we have $\sigma = 0$. For this "critical' case the gas inside the exponential horn moves in unison—the flow field has no x dependence. When $\lambda\mu > 2$—when the wave length is sufficiently long—the x-dependence of u' becomes exponential! In practice, the solution decays exponentially away from the source of the disturbance. In other words, the "acoustic" disturbance will be exponentially weak far away from the source of disturbance.

All of the above discussions assumed that exponential horn to be inifinitely long in the $+x$ direction, and the disturbance is created at $x = 0$. To properly solve the mathematical problem, we need to worry a bit on the "far field radiation" boundary condition—is there any disturbance being created at $x = +\infty$ to be accounted for?

1.1 Exercise

- What happens if there is a piston at $x = 0$ which is oscillating at very low frequency (*e.g.* one cycle per minute)? When would you expect the quasi-steady approximation to be applicable? Can you convince yourself that the two solutions agree in this limit?

- Knowledge of complex variables is assumed in this presentation. If you need help, let me know. While it is possible to deal with this problem without using complex variable, it is much easier by using it.