

# Notes #9

## MAE 533, Fluid Mechanics

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### 1 Ray Tracing

The equation of acoustics in an originally quiescent 3-dimensional medium is the classical constant coefficient linear wave equation:

$$\frac{1}{a_o^2} \frac{\partial^2 p'}{\partial t^2} = \nabla^2 p' \quad (1)$$

where  $\nabla \equiv \nabla \cdot \nabla$  is the Laplacian operator.

Now consider high frequency acoustic waves. We define two new dependent variables to replace the original single  $p'$ :

$$p' = B(\mathbf{x}, t) \exp(ik(a_o t + \phi(\mathbf{x}, t))) \quad (2)$$

where  $B(\mathbf{x}, t)$  and  $\phi(\mathbf{x}, t)$  are the new variables,  $i$  is the imaginary number ( $\sqrt{-1}$ ), and  $k$  is a constant with physical unit of reciprocal length. The frequency of the wave  $\omega$  is related to  $k$  by  $\omega \equiv k a_o$ . Hence, for given  $a_o$ , high frequency waves means  $k \gg 1$ .

Physically,  $B$  is the *amplitude* and  $\phi$  is the *phase* of the sinusoidal wave. Substituting (2) into (1), we obtain one single PDE for  $\phi$  and  $B$ :

$$(\nabla \phi)^2 = \left(1 + \frac{1}{a_o} \frac{\partial \phi}{\partial t}\right)^2 + \frac{i}{k} \left( \frac{1}{B^2} \nabla \cdot (B^2 \nabla \phi) - \frac{1}{a_o^2} \frac{\partial^2 \phi}{\partial t^2} \right) + R \quad (3)$$

where

$$R = -\frac{2i}{ka_o^2 B} \left( a_o + \frac{\partial \phi}{\partial t} \right) \frac{\partial B}{\partial t} + \frac{1}{k^2 B} \left[ \frac{1}{a_o^2} \frac{\partial^2 B}{\partial t^2} - \nabla^2 B \right] \quad (4)$$

No approximation had been made from (1), which is linear, to (3), which is nonlinear (but *quasi*-linear).<sup>1</sup> Equation (3) is valid for any value of  $k$ .

Since *two* new dependence variables were introduced to replace the original single dependent variable, we are entitled to make choose, for our convenience, a supplemental relation. We choose to restrict  $\phi$  to be real and time independent—so that its interpretation of being a phase is always valid. In fact, we can choose:

$$(\nabla \phi)^2 = 1, \quad (5)$$

so the  $B$  is governed by

$$\nabla \cdot (B^2 \nabla \phi) = B^2 ikR, \quad (6)$$

where  $R$  is now somewhat simpler because  $\phi(\mathbf{x})$  has not time dependence.

Now we consider the case when  $k$  is an asymptotically large number. Physically, this means the wave length of the acoustic waves are much smaller than the characteristics length of the problem. Equations (1) and (3) are exceedingly difficult to be solved “exactly” in the large  $k$  limit. We now further assume that  $B(x, t)$  depend “weakly” on  $t$ . How weak is weak? We assume the logarithmic time derivative of  $B(x, t)$  are  $O(1/k)$ . In the large  $k$  limit (and therefore the  $t$ -dependence of  $\phi$  and  $B$  are weak), we have  $R = O(k^{-2})$ —under the assumption that  $B$  is smooth in space and weakly time dependent. The right hand side of (6) is thus  $O(k^{-1})$  and is therefore small in the large  $k$  limit. Hence, the leading order governing PDE for  $B(x, t)$  is:

$$\nabla \cdot (B^2 \nabla \phi) \approx 0, \quad (7)$$

a wonderfully simple first order PDE. The above presentation provide the theoretical foundation of the well-known Huygens’s Principle of linear wave propagations.

In the modern age of computational fluid dynamics, (6) can be solved iteratively by the following iteration procedure when  $k$  is finite:

$$\nabla \cdot (B_{n+1}^2 \nabla \phi) = B_n^2 ikR_n, \quad (8)$$

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<sup>1</sup>A differential equation is quasi-linear if its highest order derivative(s) appear linearly in the representation.

using the  $R_o$  as the zeroth-iterant ( $\phi_o$ ). In the above discourse, subscript  $n$  represents the  $n$ -th iterant in the iteration computations.

When  $a_o$  is not a constant but is non-uniform in space, the original governing PDE contains extra terms. The methodology of ray tracing remains essentially the same. For such problems,  $\phi$  is governed by the generic non-linear first order PDE

$$(\nabla\phi)^2 = f(\mathbf{x}, t) \tag{9}$$

which can be straightforwardly computed—using an intuitively pleasing procedure which is commonly known as the Huygen’s Principle. The algebraic equation  $\phi(\mathbf{x}, t)=\text{constant}$  at any fixed time yields a *phase surface* for each specified constant, and the lines which are orthogonal to all the phase surfaces at any moment of time are called rays. The computations procedures is generally referred to as *ray tracing*. We can think about *ray tubes*, which are tubes whose walls are rays. It is then easy to conclude from inspection of (7) that the amplitude  $B$  is inversely proportional to the square root of the cross-sectional area  $A$  of the ray tube.

The mathematics of ray tracing is used in oil explorations, ultrasonic diagnostics, optics, and many other practically useful applications.

## 1.1 Exercises

1. Convince yourself that “phase” and “amplitude” are good names for  $\phi$  and  $B$ .
2. By inspection of (2), convince yourself that the frequency of the acoustic waves is  $ka_o$  in units of radians per second.
3. Apply the ray tracing method for the one-dimensional exponential horn problem. Do the results of the two theories agree?