MAE 222

Supplementary Notes

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Essential Elements of the Prandtl Boundary Layer Theory

We begin by declaring that we are interested in a large Reynolds number flow over a streamlined body, and $R_e = \rho UL/\mu$ where U and L are characteristic velocity and length, respectively.

- We begin by saying that the viscous effects are important only in a "thin" layer adjacent to the surface of the solid body. Let δ denote the characteristic thickness of this thin layer which we shall call the boundary layer. We claim that δ/L is a small number when R_e is a big number. Later on we will confirm this claim.

- Exploiting the above claim that the viscous effects are confined inside a thin layer, we conclude that viscous effects are negligible outside this thin layer. In other words, outside the boundary layer, we can use *inviscid theory*——we can ignore viscous effects. The boundary condition for inviscid flow at the edge of the thin boundary layer is that the fluid velocity is approximately tangent to the solid surface (we do not apply the no-slip condition in inviscid flow)---or, the inviscid fluid velocity normal to the solid surface should be approximately zero. When the inviscid problem is solved (somehow), we can compute the pressure (with the help of the Bernoulli's equation) anywhere in the flow field. In particular, we can find the pressure at the edge of the thin boundary layer.

- We can now focus our attention inside the viscous boundary layer. Since the boundary layer is very thin, and the fluid velocity inside the boundary layer is expected to flow essentially parallel to the tangent of the solid surface, we conclude that the pressure inside the thin boundary layer essentially do not change across the thin layer, and therefore can be approximated by the pressure at the edge of the boundary layer——where it can readily be computed from an inviscid theory as explained before.

- We now need the governing equations for the velocity profiles u(x,y) inside the thin boundary layer. Even for the boundary layer adjacent to a curved surface, we claim that the boundary layer looks flat and Cartesian——for the same reason that the earth looks flat and Cartesian to us when the characteristic length of the physical domain of interest is small in comparison to the radius of the earth. Hence, we can use the Cartesian form of the Navier-Stokes equations as a starting point in our search for the governing equations of the boundary layer, where x is the "streamwise" and y is the "normal" boundary layer coordinates, u is velocity component in the x-direction, and v is the velocity component in the y-direction. Remember, the pressure inside the boundary layer is already known. All we need now is the velocity profile (so that we can compute the wall friction).

- It turns out that we only need the streamwise momentum equation (we can ignore the momentum equation in the direction perpendicular to the surface). We find that the viscous surface stress contribution to this equation consists of two additive terms, one being the second derivative of u with respect to x (the streamwise coordinate), the other being the second derivative of u with respect to y (the normal coordinate)——for laminar flow. The contribution of the first term can be neglected.

- Prandtl's boundary layer equation is given by eqs.(I) and (II) below.

- We shall show in class that, for laminar flow, $\delta/L=O(R_e^{-1/2})$, confirming our original assertion that δ/L is small when R_e is large.

Derivation of the Momentum Integral Equation

The derivation of Prandtl's Momentum Integral equation for the boundary layer given in YMO is limited to the case of zero pressure gradient. Here is a version for arbitrary pressure gradient. The central issue is how to do a momentum balance of the boundary layer without being very precise on what we mean by the edge of the boundary layer.

We begin with the boundary layer equation in the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{(I)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \qquad \text{(II)}$$

where U(x) is the given "edge of boundary layer" u-velocity, and τ is the shearing stress. For laminar boundary layers, we have

$$\tau = \mu\frac{\partial u}{\partial y}\,. \qquad \text{(Laminar)}$$

Multiplying (I) by U, and subtracting it from (II), we obtain:

$$(u-U)\frac{du}{dx} - U\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}\,. \qquad \text{(III)}$$

Using (I), we can manipulate this to become

$$\frac{\partial}{\partial x}(u(u-U)) + \frac{\partial}{\partial y}(v(u-U)) + (u-U)\frac{dU}{dx} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}\,. \qquad \text{(IV)}$$

Integrating this equation with respect to y from y=0 to some value at the edge of the boundary layer, we obtain

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\,\theta) + \delta^*\,U\frac{dU}{dx} \qquad \text{(The Prandtl Momentum Integral Eqn.)}$$

which iinvolves the two boundary layer thicknesses θ and δ^*, U(x), and the shear stress at the wall, τ_w . The "trick" of the manipulations is that we manged to have (u-U) involved in such a way that each term in (IV) goes to zero at the edge of the boundary layer. The precise location of the edge of the boundary layer is then not an issue.

Just a reminder: x is coordinate measure along the solid surface in the direction of flow, and y measures normal to it. The coordinates is really curved, but it *looks* Cartesian. And u and v are velocity components in the x and y directions, respectively.

The *displacement thickness* δ^* and the *momentum thickness* θ are defined by:

$$\delta^* \equiv \int_0^{edge} (1-\frac{u}{U})dy\ , \qquad (1)$$

$$\theta \equiv \int_0^{edge} \frac{u}{U}(1-\frac{u}{U})dy\ . \qquad (2)$$

Note: The integrand of (1) and (2) are zero at the "edge" of the boundary layer. Thus these thicknesses can be computed without knowing precisely what the value of y at the edge of the boundary layer is. The value of "y-edge" is the value of y above which u/U is always nearly unity.

If we multiply (2) by U^2 and take the x-derivative of the resulting expression, we have (using Leinitz Rule of calculus; but the integrand is zero at the upper limit and contributes nothing):

$$\frac{d}{dx}(\theta U^2) = \int_0^{edge} \frac{d}{dx}(u^2-uU)dy \qquad (3)$$

This relation is needed to convert the integral of (IV) into the form given by the Prandtl Momentum Integral equation.

What is so great about this Prandtl's momentum equation? It is ONE equation, and contains THREE unknowns: δ^*, θ and τ_{wall} . Note: U(x) is assumed to be given and is known. To get answers, we need two additional equations.

So here is an idea: assume a functional SHAPE of the velocity profile in the boundary, using τ_{wall} as a parameter. So δ^* and θ are now functions of τ_{wall}. Now we have one equation and one unknown. This idea works pretty well for the flat plate problem!

But most importantly, we recognize that (I) and (II) is also valid for turbulenet boundary layers——provided we do not use (Laminar) expression for the shear stress at the wall. We shall see in class that it is possible to use some empirical data from pipe flow experiments to come up with a good empirical formula for turbulent problems.