A New Approach to Adaptive Nonlinear Controls*

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An accurate and realistic mathematical model is usually considered essential in controller design. The present paper presents a new approach which permits the controller to *approximately* accomplish the desired control objectives *without* the need for detailed knowledge of a mathematical model of the physical system—provided reliable and accurate sensor measurements of the output variables *and* their time derivatives are available.

Keywords: adaptive, robust, nonlinear, control, quasi-steady approximation.

1 INTRODUCTION

Consider the general dynamical model of an engineering system [1, 2, 3, 4]:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f} + \sum_{i=1}^{I} \mathbf{b}_i u^i, \tag{1}$$

where the unknown $\mathbf{x}$ is a N-dimensional column vector representing the state of the system. The first term on the right hand side of (1), $\mathbf{f}(\mathbf{x}, t)$, represents the intrinsic (*i.e.* open-loop) dynamics of the system, and the second term represents the resultant effects of I actuators of the system.

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We assume that I sensors measurements are available. The goal is: to find "control laws" for the I actuator control inputs u^i's such that the resulting I sensor measurements of the system outputs $y^i(t)$'s approximately honor certain user-specified *constraints*. The present approach exploits three new ideas:

- Instead of user-specified *algebraic* constraints, user-specified *ODE-based* constraints are imposed on the y^i's.
- Instead of requiring the ODE-based constraints to be honored exactly, finite but small errors are considered acceptable.
- Instead of looking for *static* control laws which strive to related the u^i's algebraically with $\mathbf{x}$ (through full-state feedback) or $\mathbf{y}$ (through output feedback) and t, we look for *dynamic* control laws. Most importantly, we exploit the availability of not only the measured $y^i(t)$'s, but also the availability of their time rate of change $dy^i(t)/dt$'s.

The confluence of these ideas allows the control problem to be solved *without* the need for detailed knowledge of $\mathbf{f}(\mathbf{x}, t)$—provided the control task to be performed is reasonable and the controller has ample computational power. The methodology was first advocated by Lam [7, 5, 6], and relies heavily on the insights provided by the "quasi-steady approximation," a concept familiar in reduced chemistry modeling [8, 9] of complex reaction systems.

2 FORMULATION

Each sensor signal y^i is a measurement of the state of the system:

$$y^i = \Psi^i(\mathbf{x}, t), \qquad i = 1, \ldots, I, \tag{2}$$

where the $\Psi^i(\mathbf{x}, t)$'s are assumed to be differentiable functions of its arguments. For reasons which will become clear later, they are strictly forbidden to depend on the u^i's. If $N - I > 0$, $K = N - I$ additional y^{I+k}'s can be introduced by

$$y^{I+k} = \Psi^{I+k}(\mathbf{x}, t), \quad k = 1, \ldots, K, \tag{3}$$

where the $\Psi^{I+k}(\mathbf{x}, t)$'s are chosen such that $\mathbf{y} = \{y^1, \ldots, y^N\}$ is an alternative state vector of the system (*i.e.* the Jacobian of the coordinate transformation is non-singular). We call the first I elements of $\mathbf{y}$ the *primary output* variables. and the remaining K elements the *residual* variables. We shall see later (§4) that for certain problems some residual variables need to be promoted to become *derived output* variables.

2.1 Dynamics of the Output Variables

Differentiating y^i with respect to t, we obtain

$$\frac{dy^i}{dt} = g^i + \sum_{i'=1}^{I} B^i_{i'} u^{i'}, \tag{4}$$

where

$$g^i(\mathbf{x}, t) \equiv \frac{\partial \Psi^i}{\partial t} + \mathbf{c}^i \cdot \mathbf{f}, \tag{5}$$

$$\mathbf{c}^i(\mathbf{x}, t) \equiv \frac{\partial \Psi^i}{\partial \mathbf{x}}, \tag{6}$$

$$B^i_{i'} \equiv \mathbf{c}^i \cdot \mathbf{b}_{i'}. \tag{7}$$

Since the Ψ^i's are known to be independent of the u^i's, the g^i's are independent of time derivatives of the u^i's.

In principle, $B^i_{i'}(t)$ can be directly measured on-the-fly—according to (4)—by the controller itself by pulsing the u^i's. Hence $B^i_{i'}$ is called the *pulse-response* matrix. In the following developments, we shall assume that $B^i_{i'}(t)$ is known.

2.2 ODE-Based Constraints

Normally, a conventional control problem imposes algebra-based constraints on the y^i's. For example, a tracking problem would impose $y^i = \varpi^i(t)$ where the $\varpi^i(t)$'s are the desired trajectories. Instead of such algebraic constraints, we impose ODE-based constraints in the following form:

$$\Phi^i(\dot{y}^i, \phi^i, t) \equiv \frac{dy^i}{dt} + \phi^i(\mathbf{y}, t; \varpi^{i'}) = O(\delta), \tag{8}$$

where the $\phi^i(\mathbf{y}, t; \varpi^{i'})$'s are $O(1)$ functions of their arguments and are free to be chosen by the control engineers. For example, a possible choice is:

$$\phi^i(\mathbf{y}, t; \varpi^{i'}) = \frac{1}{\tau} \sum_{i'=1}^{I} \Lambda^i_{i'} \left(y^{i'} - \varpi^{i'} \right), \tag{9}$$

where $\tau > 0$ is a time constant, $\Lambda^i_{i'}$ is a real matrix and the ϖ^i's are allowed to depend on t and perhaps the y^{I+k}'s, but not the y^i's. For example, if we choose $\Lambda^i_{i'}$ to be positive-definite and the ϖ^i's to be zeros, then this ODE-constraint is equivalent to a regulation problem which would tolerate $O(\delta)$ errors for $t >> \tau$. For reasons which shall become clear later (§5), we call the $\varpi^i(t; y^{I+k})$'s in either (8) or (9) the *residual controls* (instead of "desired trajectories").

3 WHEN $B^i_{i'}$ IS NON-SINGULAR

The exact static control law $u^i = u^i_\infty(\mathbf{x}, t)$ required to precisely honor the ODE-based constraint (8) can readily be found analytically as follows. Eliminating dy^i/dt between (4) and (8) and setting $\delta = 0$ and $u^i = u^i_\infty$, we obtain:

$$\sum_{i'=1}^{I} B^i_{i'} u^{i'}_\infty = -(\phi^i + g^i). \tag{10}$$

Assuming $B^i_{i'}$ to be non-singular, we can solve for u^i_∞ directly to obtain:

$$u^i_\infty(\mathbf{x}, t) = -\sum_{i'=1}^{I} [B^{i'}_i]^{-1}(\phi^i + g^i). \tag{11}$$

The use of this static control law $u^i = u^i_\infty(\mathbf{x}, t)$ requires detailed knowledge of the system model *and* full state feedback (specifically, the knowledge to evaluate the $g^i(\mathbf{x}, t)$'s). Since both are assumed unavailable to the controller, this exact static control is not useful.

Consider now the following ODE for u^i:

$$\frac{du^i}{dt} = -\frac{1}{\Delta t} \sum_{i'=1}^{I} W^i_{i'}(u^{i'} - u^{i'}_\infty), \tag{12}$$

where $W^i_{i'}$ is any positive-definite real matrix (with $O(1)$ eigenvalues), and Δt is a small time constant—both are to be chosen by the control engineers. Note that $u^i_\infty(\mathbf{x}, t)$ as given by (11) is known to have no dependence at all on the u^i's or their time derivatives. In the $\Delta t << 1$ limit, the quasi-steady approximation can be applied to (12)—provided u^i_∞ is smooth—yielding for $t >> \Delta t$:

$$u^i = u^i_\infty(\mathbf{x}, t) + O(\Delta t), \tag{13}$$

In other words, (12) automatically recovers an approximation to the exact static control law for $t >> \Delta t$. Any "reasonable" initial condition for $u^i(0)$ can be used. The desired accuracy can be achieved by choosing $\Delta t = O(\delta)$. Nevertheless, (12) is no more useful than (11) since the controller is unable to evaluate $u^i_\infty(\mathbf{x}, t)$.

However, we can "convert" the right hand side of (12) into a more amenable form. Assuming $B^i_{i'}$ to be non-singular, we have

$$\sum_{i"=1}^{I} [B^{i"}_{i}]^{-1} B^{i"}_{i'} = \delta^i_{i'} \tag{14}$$

where $\delta^i_{i'}$ is the identify matrix. We now rewrite $W^i_{i'}$ as follows:

$$W^i_{i'} = \sum_{i"=1}^{I} W^i_{i"} \delta^{i"}_{i'}. \tag{15}$$

Using (15) in (12) and with the help of (14), (10) and (4), we obtain,

$$\frac{du^i}{dt} = -\frac{1}{\Delta t} \sum_{i'=1}^{I} Z^i_{i'} (\frac{dy^{i'}}{dt} + \phi^{i'}), \tag{16}$$

where $Z^i_{i'}$ is given by:

$$Z^i_{i'} \equiv \sum_{i"=1}^{I} W^i_{i"} [B^{i'}_{i"}]^{-1}. \tag{17}$$

Equation (16) is the proposed dynamic control law when $B^i_{i'}$ is non-singular.

Since (16) is *mathematically identical* to (12) which is known to be stable in the small Δt limit, we conclude that (16) is stable there also. In essence, the microprocessor-based controller is asked to numerically integrate (16), leaving Nature the task of analog integration of (1). To perform the numerical computations, the controller requires the following:

- the selection of the smallest possible $\Delta t = O(\delta)$ consistent with practicality,
- adequate knowledge of $B^i_{i'}(t)$ to choose $Z^i_{i'}$ so that it is consistent with a positive-definite $W^i_{i'}$,

- the measured output variables $y^i(t)$'s—plus a decision on what to do with the residual controls ϖ^i's—so that the $\phi^i(t, \mathbf{y}; \varpi^{i'})$'s can be evaluated, and
- the $dy^i(t)/dt$'s.

If $B^i_{i'}(t)$ is known, then the simple-minded choice of $Z^i_{i'} = [B^{i'}_i]^{-1}$ would do very well indeed. In general, it is preferred that both the $y^i(t)$'s and the $dy^i(t)/dt$'s be directly measured with good signal-to-noise ratios. Only when the $y^i(t)$'s have sufficiently good signal-to-noise ratios should the $dy^i(t)/dt$'s be computed by numerical differentiation.

4 WHEN $B^i_{i'}$ IS SINGULAR OR NEARLY SINGULAR

When u^i_∞ as computed from (11) either does not exist or is unacceptably large, the $B^i_{i'}$ is said to be either singular or nearly singular. Many engineering systems have singular or nearly-singular $B^i_{i'}$'s. Generalizations to include such problems involves the extensive use of the concepts and tools of singular value decomposition [10]. The singular value decomposition of $B^i_{i'}$ is written as follows:

$$B^i_{i'} = \sum_{i''=1}^{I} [U^{i''}_i]^T \omega(i'') V^{i''}_{i'}, \tag{18}$$

where $U^{i''}_i$ and $V^{i''}_{i'}$ are unitary matrices and $\omega(i'') \geq 0$ are the singular values ordered in descending magnitudes. The number of non-zero singular values is by definition the traditional rank of $B^i_{i'}$. To treat singular and nearly-singular problems in a unified manner, we introduce $|\epsilon|$, a "small" number to serve as the *singular value threshold*. The number of singular values at or below threshold is denoted by J. The *epsilon-rank* of a matrix is defined as the number of above threshold singular values. The bottom J rows of $U^i_{i'}$, the U^{I+1-j}_i's, play a major of role in the subsequent developments. They are called the *left epsilon-vectors* of $B^i_{i'}$ because of their ability to nearly (or completely) annihilate $B^i_{i'}$:

$$\sum_{i=1}^{I} U^{I+1-j}_i B^i_{i'} = \omega(I+1-j) V^{I+1-j}_{i'} = O(\epsilon), \qquad j = 1, \ldots, J. \tag{19}$$

In order for a control problem with $J > 0$ to have a solution, J additional *derived output variables* y^{I+j}'s must be found such that the epsilon-rank of the resulting extended $M \times I$ rectangular matrix B_i^m (where $M = I + J$) is I. For the sake of simplicity, we shall adopt the following special ODE-based constraints for the M output variables:

$$\Phi^m = \frac{dy^m}{dt} + \frac{1}{\tau}\sum_{m'=1}^{M} \Omega^m_{m'}(y^{m'} - \varpi^{m'}) = O(\delta), \qquad m = 1, \ldots, M, \tag{20}$$

which is in the form of (8) and (9). However, whenever $M > I$, certain elements of $\Omega^m_{m'}$ and some of the ϖ^m's are not free to be user-specified as we shall show immediately below. Only the theory for the case $J = 1$ (and thus $M = I + 1$) is presented here (the general theory will be presented in a separate paper [11]). The matrix $B^i_{i'}$ here is thus either singular or nearly singular because of one of its singular value is at or below threshold, and its lone left epsilon-vector is U^I_i.

For the $J = 1$ case, the lone *derived output variable* y^{I+1} is defined by:

$$\sum_{i=1}^{I} U^I_i \Phi^i = 0, \tag{21}$$

where the Φ^i's were previously defined by (20), the elements of $\Omega^m_{m'}$ satisfy:

$$\Omega^i_{i'} = \Lambda^i_{i'} = \text{ free to be chosen}, \quad i, i' = 1, \ldots, I, \tag{22}$$

$$\Omega^i_{I+1} = -\tau[U^I_i]^T, \qquad i = 1, \ldots I, \tag{23}$$

and the ϖ^m's satisfy

$$\sum_{i=1}^{I}\sum_{m=1}^{I+1} U^I_i \Omega^i_m \varpi^m = 0. \tag{24}$$

where τ, $\Lambda^i_{i'}$ and Ω^{I+1}_m are completely free to be user-specified. Because of (24), only I of the $I + 1$ ϖ^m's are available to serve as residual controls.

Equation (21) can be rewritten in a more familiar form, showing explicitly that y^{I+1} is "derived" from the dy^i/dt's and the y^i's:

$$y^{I+1} = \Psi^{I+1}(\mathbf{x}, t; \epsilon) \equiv \sum_{i=1}^{I} U^I_i \left(\frac{dy^i}{dt} + \frac{1}{\tau}\sum_{i'=1}^{I} \Lambda^i_{i'} y^{i'} \right). \tag{25}$$

Note that when the dy^i/dt's in the above equation are eliminated using (4), the resulting $\Psi^{I+1}(\mathbf{x},t)$'s has an $O(\epsilon)$ dependence on the u^i's. This is in stark contrast to the primary $\Psi^i(\mathbf{x},t)$'s which are strictly not allowed to have such dependence at all. This slight u^i dependence has profound consequences in the subsequent theoretical developments.

The ODE for y^{I+1} is derived by differentiating $\Psi^{I+1}(\mathbf{x},t;\epsilon)$ with respect to time, and is in the form of (4). We assume that the epsilon-rank of the resulting extended $M \times I$ matrix $B_i^m(t)$ is I. The *epsilon-inverse* of the extended B_i^m is denoted by $[B_i^m]^+$ and is given by [10]:

$$[B_i^m]^+ \equiv \sum_{i'=1}^{I} [\check{V}_i^{i'}]^T \frac{1}{\check{\omega}(i')} \check{U}_m^{i'}, \tag{26}$$

where $\check{U}_m^i$, $\check{V}_{i'}^i$ and $\check{\omega}(i) > \epsilon$ are the unitary matrices and the singular values of the extended B_i^m. Note that g^{I+1} now contains an $O(\epsilon)$ term involving the du^i/dt's.

We are now ready to propose the dynamic control law:

$$\frac{du^i}{dt} = -\frac{1}{\Delta t} \sum_{m=1}^{I+1} \check{Z}_m^i \left(\frac{dy^m}{dt} + \frac{1}{\tau} \sum_{m'=1}^{I+1} \Omega_{m'}^m (y^{m'} - \varpi^{m'}) \right), \tag{27}$$

where

$$\check{Z}_m^i \equiv \sum_{i'=1}^{I} W_{i'}^i [B_{i'}^m]^+. \tag{28}$$

As before, $\check{Z}_m^i$ is to be chosen so that $W_{i'}^i$ is positive-definite.

The choice of Δt now requires special attention. If the original $B_{i'}^i$ is strictly singular (*i.e.* $\epsilon = 0$), then (27) as an ODE can be shown to be always stable in the limit of $\Delta t \to 0$. Any standard integration algorithm can then be used to perform the numerical integration, and there is no theoretical lower limit to the value of Δt. However, if the original $B_{i'}^i$ is nearly-singular (*i.e.* $\epsilon \neq 0$), then the right hand side of (27) has a "hidden" $\epsilon du^i/dt$ term, and the stability of (27) in the small Δt limit is not clear in general. We shall show via an example in the next section that a lower limit $\Delta t > O(\epsilon)$ must then be imposed to ensure stability. It can be shown that if the original $B_{i'}^i$ had more than one below threshold singular values ($J \geq 2$), then the resulting right hand side of (27) would have hidden terms in the form of an $O(\epsilon)$ factor times the α-th order time derivatives of u^i where $\alpha \leq J$.

For $\alpha \geq 2$, the corresponding exact (27) could be intrinsically unstable when it is being treated strictly as an ODE, and the instability cannot be cured by adjusting Δt. However, various artifices can be used to render impotent these troublesome $O(\epsilon)$ time derivatives terms. For example, (27) when converted to finite difference form can be stabilized using numerical damping, or the values of the u^i's could be held momentarily constant while the output variables are being measured. Such artifices are currently under investigation and the results will be presented in a separate paper [11].

5 AN EXAMPLE

In this section, all superscripts are moved downward to become subscripts to avoid confusion with exponents. Consider a problem with $N = 3$, $I = 1$ and:

$$\mathbf{f} = \begin{vmatrix} f_1(x_1, x_2, x_3, t) \\ f_2(x_1, x_2, x_3, t) \\ f_3(x_1, x_2, x_3, t) \end{vmatrix}, \qquad \mathbf{b}_1 = \begin{vmatrix} \epsilon \\ 1 \\ 0 \end{vmatrix}, \qquad y_1 = x_1. \tag{29}$$

We assume t, y_1 and u_1 are $O(1)$ (by appropriate non-dimensionalization), $\partial f_1/\partial x_2$ is $O(1)$ and is never zero, and ϵ is a small parameter of uncertain sign. We have $B_{11} = \epsilon$ and the problem is either singular ($\epsilon = 0$) or nearly-singular ($\epsilon \neq 0$) because the $u_{1\infty}$ computed using the non-singular methodology is unacceptable. Hence we choose $|\epsilon|$ to be our singular value threshold. The left epsilon-vector of B_{11} is then simply $U_1^1 = 1$.

The control goal is to achieve $|y_1(t)| < \delta$ for $t >> \tau$ where δ is small but finite and $\tau = O(1)$—using $u^i = O(1)$. This regulation control task is expressed as an ODE-based constraints in the form of (20). We choose, in deference to (23):

$$\Omega_{mm'} = \begin{vmatrix} \Lambda_{11} & -\tau \\ \Omega_{21} & \Omega_{22} \end{vmatrix}, \tag{30}$$

where Λ_{11}, Ω_{21} and Ω_{22} are free parameters. The derived output variable y_2 is then given by:

$$y_2 = \frac{dy_1}{dt} + \frac{\Lambda_{11}}{\tau} y_1. \tag{31}$$

In deference to (24), we require:

$$\Lambda_{11}\varpi_1 - \tau\varpi_2 = 0, \tag{32}$$

so that only one residual control (either ϖ_1 or ϖ_2) is available to be exploited.

The dynamical equation expressed in the form of (4) for y_2 is readily derived by differentiation. We have:

$$\frac{dy_2}{dt} = g_2 + B_{21}u_1, \tag{33}$$

where

$$g_2 \equiv \tilde{g}_2 + \epsilon\frac{du_1}{dt}, \tag{34}$$

and

$$\tilde{g}_2(x_1, x_2, x_3, t) \equiv \frac{\partial f_1}{\partial t} + f_1\frac{\partial f_1}{\partial x_1} + f_2\frac{\partial f_1}{\partial x_2} + f_3\frac{\partial f_1}{\partial x_3}. \tag{35}$$

As mentioned previously, g_2 now contains an $O(\epsilon)$ term involving du_1/dt. The elements of the extended 2×1 pulse-response matrix B^m_1 are:

$$B_{11} = \epsilon, \quad B_{21}(t) = \frac{\partial f_1}{\partial x_2} + \epsilon(\frac{\partial f_1}{\partial x_1} + \frac{\Lambda_{11}}{\tau}). \tag{36}$$

We assume the epsilon-rank of B_{m1} is always one; *i.e.* B_{21} is $O(1)$ and never crosses zero at any time. The singular value decomposition of B_{m1} is easily found:

$$\check{U}_{1m} = \frac{[\epsilon, B_{21}]}{\sqrt{\epsilon^2 + B_{21}^2}}, \quad \check{U}_{2m} = \frac{[B_{21}, -\epsilon]}{\sqrt{\epsilon^2 + B_{21}^2}}, \quad \check{\omega}(1) = \sqrt{\epsilon^2 + B_{21}^2}, \quad \check{V}_{11} = 1. \tag{37}$$

Following (27), we have the following dynamic control law (the scalar W_{11} is taken to be unity):

$$\frac{du_1}{dt} = -\frac{1}{\Delta t}\left(\frac{B_{21}}{\epsilon^2 + B_{21}^2}\right)\left(\frac{dy_2}{dt} + \frac{\Omega_{21}}{\tau}(y_1 - \varpi_1) + \frac{\Omega_{22}}{\tau}(y_2 - \varpi_2)\right), \tag{38}$$

where Δt remains to be selected. Note that no detailed information on f_1, f_2 or f_3 are required to numerically integrate (38)—only the knowledge of the extended pulse-response matrix B_{m1} is needed.

Is (38) stable in the limit of small ϵ and small Δt? Eliminating dy_2/dt using (33), we have:

$$\frac{du_1}{dt} = -\frac{1}{\Delta t}\left(\frac{B_{21}^2}{\epsilon^2 + B_{21}^2}\right)\left(u_1 + \epsilon\frac{du_1}{dt} - \tilde{u}_{1\infty}\right), \tag{39}$$

where

$$\tilde{u}_{1\infty}(x_1, x_2, x_3, t) \equiv -\frac{1}{B_{21}}\left(\tilde{g}_2 + \frac{\Omega_{21}}{\tau}(y_1 - \varpi_1) + \frac{\Omega_{22}}{\tau}(y_2 - \varpi_2)\right). \quad (40)$$

We now see that du_1/dt appears on both sides of this equation. Since the sign of ϵ is uncertain, *the lower limit* $\Delta t > O(\epsilon)$ *must be imposed* to ensure (39)—and therefore (38)—to be always stable.

For $t >> \Delta t > O(\epsilon)$, the left hand side of (38) can be neglected (*i.e.* applying the quasi-steady approximation) to yield:

$$\frac{d^2 y_1}{dt^2} + (\frac{\Lambda_{11}}{\tau} + \frac{\Omega_{22}}{\tau})\frac{dy_1}{dt} + \left(\frac{\Omega_{21}}{\tau} + \frac{\Lambda_{11}\Omega_{22}}{\tau^2}\right)(y_1 - \varpi_1) = O(\Delta t), \quad (41)$$

where y_2 and ϖ_2 had been eliminated by the use of (31) and (32), respectively. The values for τ, Λ_{11}, Ω_{21} and Ω_{22} can be chosen so that $y_1(t)$ "tracks" $\varpi_1(t)+O(\Delta t)$ for $t >> \tau$ with the desired transient behavior—assuming that $\varpi_1(t)$ is "sufficiently slowly varying." For the regulation problem, we choose $\varpi_1 = 0$ and the resulting accuracy is $\delta = O(\Delta t) > O(\epsilon)$. In other words, an non-zero ϵ (of uncertain sign) is detrimental to the accuracy performance of the controller—when the primary output y_1 is x_1. However, it can easily be shown [11] that all the detrimental effects of an non-zero constant ϵ can be removed if the primary output y_1 is changed from x_1 to $x_1 - \epsilon x_2$.

5.1 Influencing Zero Dynamics

What about the "zero dynamics" of the residual variables? Obviously, some additional information must be provided to the controller in order for it to be even aware of the existence of the residual variables. Consider now the special case:

$$f_1 = x_2, \quad (42)$$

$$f_2 = \text{arbitrary functions of } x_1, x_2, x_3 \text{ and } t, \quad (43)$$

$$f_3 = \kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3 + \xi, \quad (44)$$

where the κ_n's are $O(1)$ non-zero constants of uncertain sign, and $\xi(t)$ is a slowly varying function of t. We introduce the residual variable:

$$y_3 = x_3, \quad (45)$$

and assumed that it is also measured and made available. Using (42) in the second element of (1), we obtain $x_2 = dy_1/dt - \epsilon u_1$. Eliminating the x_n's in favor of the y_n's from (44), we obtain the following governing ODE for y_3:

$$\frac{dy_3}{dt} = \kappa_1 y_1 + \kappa_2 (\frac{dy_1}{dt} - \epsilon u_1) + \kappa_3 y_3 + \xi. \tag{46}$$

If y_1 and u_1 are considered known functions of t, then the stability of (46) depends completely on the sign of κ_3. If $\kappa_3 < 0$, the zero dynamics problem is stable and such problems are usually said to be "minimum phase." If $\kappa_3 > 0$, then the problem is "non-minimum phase," and the possibility of using the residual control $\varpi_2(t; y_3)$ to exert some (non-unique) influence on (46) can be explored. One such possibility is to choose:

$$\varpi_1 = \lambda y_3, \tag{47}$$

where λ is a constant to be determined. Equations (20) and (46) are now three coupled simultaneous ODE's for y_1, y_2 and y_3 for $t >> \Delta t$. Assuming y_3 to be slowly varying, we have from (41) $y_1 \approx \lambda y_3$ for $t >> \tau$. Eliminating y_1 from (46) using this approximation, we have:

$$\frac{dy_3}{dt} = -\sigma^2 y_3 - \frac{1}{\lambda\kappa_2 - 1}\left(\epsilon\kappa_2 u_1 - \xi\right), \tag{48}$$

where

$$\sigma^2 \equiv \frac{\lambda\kappa_1 + \kappa_3}{\lambda\kappa_2 - 1}. \tag{49}$$

Solving for λ, we obtain:

$$\lambda = \frac{\sigma^2 + \kappa_3}{\sigma^2\kappa_2 - \kappa_1}. \tag{50}$$

Whenever this λ is used in (47), y_3 will be stabilized—but at the cost of abandoning the goal of holding $y_1 = O(\delta)$. Only when $\xi(t)$ is $O(\epsilon)$ will y_3 decay to $O(\epsilon)$, and therefore so will ϖ_1 and y_1. To be consistent with the assumption that $y_3(t)$ be slowing varying, the value of σ chosen should be moderately small. Summarizing, the dynamics of the residual variable y_3 has been stabilized (for this specific example problem) by exploiting the residual control ϖ_1—when additional information is available.

The readers can easily verify the validity of all the above claims by direct numerical simulations using desktop computers.

6 DISCUSSIONS

The present theory requires reliable sensor signals with sufficiently good signal-to-noise ratios be available. It strongly prefers direct measurements of the time derivatives of the output variables instead of getting them by numerical differentiation. In addition, the pulse-response matrix $B_i^m(t)$ is also assumed known. If the choice and design of the sensors and actuators are at the disposal of the control engineers, a time-invariant non-singular $B_{i'}^m$ is obviously preferred. If B_i^m is time dependent and is unknown, we assume that it can be directly measured on-the-fly by pulsing the u^i's.

When $B_{i'}^i$ is singular or nearly singular, additional derived output variables must be found so that the epsilon-rank of the extended $M \times I$ pulse-response matrix B_i^m is I. If no such pulse-response matrix can be found, or if the computed $u^i(t)$'s are found to be too large to be acceptable, then the control task to be performed is probably an unreasonable task for the system in question.

Because the controller has essentially no information about the system, it must totally trusts the sensors measurements—it is inherently unable to distinguish between measurement noise and actual physical perturbations to the system, or to surmise that perhaps some or all of the sensor measurements are wrong. If "low frequency" random noises in the measurements are present, the controller will suppress them, causing errors in the actual performance of the system. If "very high frequency" random noises in the measurements are present, the value of Δt can be increased somewhat to filter them out, again at the cost of degrading the performance. Fundamentally, reliable and accurate measurements of the output variables are crucial to the success of the present controller. Obviously, if some detailed information about the system is available, one should be able to take advantage of it to reduce the total reliance on the sensor measurements (and the need for their time derivative(s)).

As it stands, the controller assumes the (unobserved) zero dynamics is satisfactory. If some influence on the zero dynamics is desired, certain detailed knowledge of the mathematical model of the system must be provided. Whether a beneficial influence can be achieved by exploiting the residual controls must be dealt with on a case by case basis—using the detailed knowledge provided.

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