Noise Tolerance of the Universal Dynamic Control Law

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Abstract

Both the Universal Dynamic Control Law (UDCL)—advocated by this author—and the classical High Gain Proportional-Integral Control Law (HGPICL) are robust controllers in the sense that they have the ability to approximately honor user-specified control goals without the need of detailed knowledge of the system in question. This paper studies the impacts of measurement noise, and shows that the UDCL is substantially superior to the HGPICL in its tolerance of measurement noise. Numerical simulations are presented to support the theoretical conclusions.

1 Introduction

In a series of recent papers, Lam [1, 2, 3] advocated the Universal Dynamic Control Law (UDCL) which main merit is that it can approximately honor control goals without the need for detailed knowledge of the system in question. This feat is made possible by the requirement that good quality real time measurements of the time derivative(s) of the output variables be provided—in addition to the output variables themselves. It is well known that the classical High Gain Proportional-Integral Control Law (HGPICL) can also achieve this feat (in the limit of very high gain) without the need of real time measurements of the time derivatives. However, when there is noise in the output variable measurements, the poor performance of the HGPICL is notorious. The present paper analyzes the impacts of measurement noise on both of these two control laws. Results of numerical simulations are presented in support of the theoretical conclusions.

2 Statement of the Control Problem

Given a dynamical system with $I$ actuators:

$$\frac{dx}{dt} = f + \sum_{i=1}^{I} b_i u^i$$

where $x$ is a $N$-dimensional state vector, $f(x, t)$ represents the dynamics of the system, and the $b_i(x, t)$'s and $u^i$'s represent the directions and amplitudes of the control forces exerted by the each of the $I$ actuators. Without loss of generality, we assume the $u^i$'s are $O(1)$ by appropriate normalization. There are $I$ output variables of special interest to the control engineers:

$$y^i = \psi^i(x, t), \quad i = 1, \ldots, I.$$ (2)

The $y^i$'s are also assumed to have been appropriately normalized. The control problem is to find $u^i(t)$’s such that the following $I$ user-specified ODE-based constraints are approximately satisfied:

$$\frac{dy^i}{dt} + \phi^i(y, t) = O(\epsilon), \quad i = 1, \ldots, I,$$ (3)

where the $\phi^i(y, t)$’s are user-specified functions and $\epsilon$ is a small number representing the threshold of acceptable errors. The challenge is to honor the control goal eq.(3) without the need for detailed information about the system (such as $f(x, t)$ and the $\psi^i(x, t)$’s). It is assumed that the real time sensor measurements needed by the controller black box are of “sufficiently good quality.” Most importantly, the “sampling time” $t_s$—the time needed by the controller black box to update the $u^i(t)$’s—is assumed to be a very small number in comparison to the system characteristic time which is assumed to be $O(1)$.

The ODEs governing the $y^i$’s can readily be obtained by differentiating eq.(2) with respect to time:

$$\frac{dy^i}{dt} = g^i + \sum_{i' = 1}^{I} B^i_{i'} u^{i'},$$

where

$$g^i = \frac{\partial \psi^i}{\partial t} + \frac{\partial \psi^i}{\partial x} \cdot f.$$ (5)
\[ B^i_k = \frac{\partial \psi^i}{\partial x} \cdot b^i. \] (6)

The matrix \( B^i_k \) is called the pulse-response matrix of the system in question because it represents the “jump” of \( y^i \) in response to an unit pulse of \( u^i \). Hence, it can be (diagonically) measured by the controller black box on-the-fly, at least in principle. In the present paper, we shall limit our attention to the so-called “regular case,” i.e. \( B^i_k(t) \) is assumed nonsingular—all of its \( I \) singular values are \( O(1) \).

3 Sensor Measurements

Since sensor measurements always involves noise, they are marked by asterisk subscripts to distinguish them from their actual values. We have:

\[
\begin{align*}
\dot{y}^i_k(t) &= y^i(t) + \delta^i_k(t), & (7) \\
\ddot{y}^i_k(t) &= \dot{y}^i(t) + \delta^i_2(t), & (8) \\
\dddot{y}^i_k(t) &= \ddot{y}^i(t) + \delta^i_3(t), & (9) \\
& \quad \ldots
\end{align*}
\]

where the \( \delta^i_k(t) \)'s denote measurement noises (the subscript \( k \) indicate the order of derivative involved). We assume that time derivative data are directly measured—numerical differentiation is strongly frown upon. The sensor data is said to be of “sufficiently good quality” when \( \delta^i_k(t) \)'s are \( O(\epsilon) \) (for the needed \( k \)'s). No assumption that they have zero mean is required.

4 High Gain Integral Control Law (HGPICL)

Consider now the simple-minded static control law \( u^i = u^i_o(y^*_o, t) \) where:

\[
u^i_o(y^*_o, t) = -\frac{1}{\mu t_s} \sum_{i'=1}^{I} S^i_k(t) \left\{ y^{i'}(t) + \delta^{i'}(t) \right\} + \int_0^t \phi^{i'}(y(t), t) dt + C^{i'}\}
\]

where \( t_s \) is the sampling time of the controller black box as mentioned previously, the \( C^{i'} \)'s are any reasonable constants—they can be used to specify the initial condition for \( u^i(0) \). The matrix \( S^i_k \) is defined by:

\[
S^i_k = \sum_{i'=1}^{I} [B^i_{k'}]^{-1} W^i_{i'}, \]

where \( W^i_{i'} \) is any matrix whose eigenvalues all have \( O(1) \) positive real parts. The “design” of the HGPICL thus consists of (1) choosing the value of \( \mu = O(1) > 0 \) and (2) finding a \( S^i_k \) such that the associated \( W^i_{i'} \) has the required property. It can easily be verified analytically that when \( u^i = u^i_o(y^*_o, t) \) is used (i.e. no measurement errors) with \( 1 >> t_s = O(\epsilon) > 0 \) in eq.(4), the control goal eq.(3) is indeed honored with \( \epsilon = O(t_s) \) for \( t >> t_s \). Because small \( t_s \) means high gain and PI feedback is used, eq.(10) is called the High Gain PI Control Law (HGPICL).

The poor performance of \( u^i = u^i_o(y^*_o, t) \) when there is measurement noise is well known. When measurement noise is present, we can rewrite \( u^i_o(y^*_o, t) \) using eq.(7) as follows:

\[
u^i_o(y^*_o, t) = u^i_o(y^*_o, t) + n^i(t) \]

where \( n^i(t) \) is defined by:

\[n^i(t) = \frac{\delta^i_0(t)}{\mu t_s} \]

and \( \delta^i_0(t) \) is approximately given by:

\[
\delta^i_0(t) \approx \delta^i_0(t) + \int_0^t \sum_{i'=1}^{I} \frac{\partial \phi^{i'}}{\partial y^{i'}(0)} \delta^{i'}(t) dt.
\]

Hence \( n^i(t) \) represents the “errors” introduced by \( \delta^i_0(t) \) to \( u^i(t) \). Note that it is singular in the small \( t_s \) limit.

5 The Universal Dynamic Control Law (UDCL)

Lam [1, 2, 3] proposed the following Universal Dynamic Control Law (UDCL):

\[
\frac{du^i}{dt} = -\frac{1}{\mu t_s} \sum_{i'=1}^{I} S^i_k(t) \left( \dot{y}^{i'}(0) + \phi^{i'}(y^*_o, t) \right)
\]

where

\[
S^i_k = \sum_{i'=1}^{I} W^i_{i'} [B^i_{k'}]^{-1},
\]

and \( W^i_{i'} \) is required to have the same eigenvalue property as stated in the previous section. Any reasonable initial condition for \( u^i(0) \) can be used.

Using eq.(7), eq.(8) and eq.(16) in eq.(15), we obtain:

\[
\frac{du^i}{dt} = -\frac{1}{\mu t_s} \sum_{i'=1}^{I} \left( \sum_{i'=1}^{I} W^i_{i'} [B^i_{k'}]^{-1} \right) \left( \dot{y}^{i'}(0) + \phi^{i'}(y^*_o, t) + \delta^i_0(t) \right)
\]

where

\[
\delta^i_0(t) \approx \delta^i_0(t) + \int_0^t \frac{\partial \phi^{i'}}{\partial y^{i'}} \delta^{i'}(t)
\]
Using eq.(4) to eliminate \( \dot{y} \) from eq.(17), we obtain:

\[
\frac{du^i}{dt} = -\frac{1}{\mu t_s} \sum_{i'=1}^I W_{i'}^i \left( u_{i'} - (u_{i'}(y,t) + \tilde{n}_{i'}(t)) \right) 
\]  \hspace{1cm} (19)

where

\[
u_{i'}(y,t) \equiv -\sum_{i'=1}^I [B^i_{i'}]^{-1} \left( g_{i'} + \phi_{i'} \right)
\]  \hspace{1cm} (20)

and \( \tilde{n}(t) \) defined by:

\[\tilde{n}(t) \equiv \sum_{i'=1}^I [B^i_{i'}]^{-1} \delta^i_{i'}(t) \]  \hspace{1cm} (21)

represents the noise-induced errors. It can easily be verified analytically that \( u^i = u_{i'}(y,t) + \tilde{n}_{i'}(t) + O(t_s) \), \hspace{1cm} (22)

Hence, any smooth \( u^i(t) \)’s generated by numerical integration of eq.(15) is guaranteed to be a good approximation to \( u_{i'}(t) \) for \( t >> \mu t_s \). Hence, the UDCL works best when \( t_s, \delta^i_{i'}(t) \) and \( \delta^i_{i'}(t) \) are small.

6 Numerical Simulations

Numerical simulations were performed on the following simple \( N = I = 1 \) problem:

\[ f(x,t) = \alpha x + \sin(\beta x) \]  \hspace{1cm} (23)

\[ + H(t-5) - H(t-15) \]  \hspace{1cm} (24)

\[ b = \text{constant} \]  \hspace{1cm} (25)

\[ y = x, \hspace{1cm} (i.e. \hspace{1cm} \psi(x) = x) \]  \hspace{1cm} (26)

\[ \phi(y) = \frac{y}{\tau} \]  \hspace{1cm} (27)

where \( H(\xi) \) denotes a unit step function located at \( \xi = 0 \). \( \alpha \) and \( \beta \) are any \( O(1) \) entities which are allowed to depend on \( t \), and \( \tau > 0 \) is a (user-chosen) timescale for the decay of \( y \). We have chosen \( f(x,t) \) to be a (contrived) nonlinear function of \( x \) (and discontinuous with respect to \( t \)) to show off the fact that the control laws under discussion do not care what it is. Since \( N = I = 1 \), the matrices \( B^i_{i} \), \( S^i_{i} \) and \( \tilde{S}^i_{i} \) are \( O(1) \) scalars.

We used \( \alpha = \cos(t) \), \( \beta = 1 \), \( b = 1 \), \( S^i_{i} = \tilde{S}^i_{i} = 1/b \) and \( \tau = 1 \) for all the simulation runs. The integration timestep used was always 0.01 which is identified with the black box sampling time \( t_s \). Measurement noise \( n(t) \) of the HGPICL controller is a random number sequence refreshed after every integration step, and the same is used for \( \tilde{n}(t) \), the measurement noise of the UDCL controller. The initial conditions used were arbitrarily chosen to be \( x(0) = 1 \) and \( u(0) = 0 \). The only “design” parameter left to be chosen in both of these two control laws is \( \mu \).

For all simulation runs, the HGPICL is used for \( 0 \leq t < 10 \), and the UDCL is used for \( 10 \leq t < 20 \). Simulation results are shown in Figs. 1 to 6. Figs. 1 and 2 show the noise-free case using \( \mu = 2 \). Both controllers performed well. The discontinuities of \( f \) at \( t = 5 \) and \( t = 15 \) were handled in strides. Figs. 3 and 4 show that results when measurement noise is present (the amplitudes of \( n(t) \) and \( \tilde{n}(t) \) used were both 0.1). The superiority of the UDCL to noise tolerance is clearly shown, as expected by the theory. The beneficial effects of using a somewhat larger \( \mu \) is shown in Figs. 5 and 6. As a practical matter, most engineering systems would find the large oscillations in the HGPICL generated \( u(t) \) unacceptable.

 Entirely similar results are obtained for other (reasonable) choices of system parameters.

7 Concluding Remarks

In the control community, measurements of time derivatives are traditionally frowned upon. Recent advances of modern sensor technology suggest that this posture should be re-examined. There is no fundamental reason to believe that measurements of time derivatives are more problematical than the output variables themselves. Once the premise that all needed real time measurements are of good quality and can be made available, the UDCL—which needs no detailed knowledge of the system—is clearly a viable option for controlling practical systems. While the UDCL can be recovered by formally differentiating the HGPICL with respect to time, the two control laws are conceptually distinct. Mathematically, the HGPICL attempts to overwhelm \( f \) by brute force, while the UDCL takes the measures of the \( g^i \)’s via the measured \( \dot{y}^i(t) \)’s, and utilizes the stiffness of eq.(19) to make \( u^i \) generated by numerical integration to approach \( u_{i'} \). The reason the UDCL is more noise tolerant is the direct result of the \( \dot{y}^i(t) \)’s data being assumed to be of good quality, and is not obtained by numerical differentiation of the \( y^i(t) \)’s.

The present paper is limited to the regular case when \( B^i_{i} \) is nonsingular (and not “nearly singular”). Similar conclusions are expected for the irregular case.

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1 All superscripts and subscripts will be omitted in this section because this is a scalar problem.
lications such as time delays and hysteresis can be handled in strides [5]. The basic idea of UDCL has also been extended to deal with a special class of optimal control problems.

References

Figure 1: $x$ versus $t$ for $\mu = 2$ (noise-free).
$\alpha = \cos(t), \beta = 1, \ b = 1, \ \tau = 1, \ t_s = 0.01$.
HGPICL: $0 \leq t < 10, \ n(t) = 0$.
UDCL: $10 \leq t < 20, \ \bar{n}(t) = 0$

Figure 2: $u$ versus $t$ for $\mu = 2$ (noise-free).
All parameters same as in Fig. 1.

Figure 3: $x$ versus $t$ for $\mu = 2$ (with noise).
HGPICL: $0 \leq t < 10, \ \text{amplitude of } n(t) = 0.1$.
UDCL: $10 \leq t < 20, \ \text{amplitude of } \bar{n}(t) = 0.1$
All parameters same as in Fig. 1.

Figure 4: $u$ versus $t$ for $\mu = 2$ (with noise).
All other parameters same as in Fig. 3.

Figure 5: $x$ versus $t$ for $\mu = 10$ (with noise).
All other parameters same as in Fig. 3.

Figure 6: $u$ versus $t$ for $\mu = 10$ (with noise).
All other parameters same as in Fig. 3.