

On Stability Theories and Predictions

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February 26, 2004

1 Introduction

We learned very early from the experiments of Osborne Reynolds and the data presented in the Moody Diagram: *laminar fluid flow is not always what actually happens under the so-called “steady flow” assumption.* Generally speaking, flows are always laminar at very small Reynolds Numbers, and are always turbulent at sufficiently large Reynolds Numbers.

So, there are two theoretical and practical questions to be addressed:

- when does the transition from laminar to turbulent flow occur?
- What do we do after the flow becomes turbulent?

Chapter 5 of White, and these few pages, deal with the first bullet.

2 The role of experiments

The Moody diagram tells us what the real world says about straight round pipes. For Reynolds Number R_e (based on diameter) less than about 2300, the flow is laminar, and the roughness of the pipe is irrelevant. For R_e above 2300, the flow is turbulent and roughness of the pipe is relevant. Most importantly, the turbulent friction is *much* higher than what the laminar theory would predict at very high Reynolds Numbers—even for very smooth

pipes. For R_e in the neighborhood of 2300, the data is somewhat erratic and scattered.

If you are interested in some other flow geometry, you would need data analogous to the Moody diagram in order to be fully informed.

3 The role of theories

3.1 Definition of stability

Let $\phi(x, y, z; t)$ be an object of our attention. If ϕ “blows up” as $t \rightarrow \infty$, then we say ϕ is unstable.

The “steady state” assumption is a very popular assumption for theorists. If we made the steady state assumption and found a theoretical solution $\phi_{ss}(x, y, z)$ to our problem, we must address the question of its stability: if the steady state solution is very slightly disturbed, will the disturbances decay or grow?

3.2 Saying it with mathematics

Let the governing equation for ϕ be represented by:

$$\mathcal{N}[\phi] = 0 \tag{1}$$

where $\mathcal{N}[\cdot]$ is some (nonlinear) differential operator (either ODEs or PDEs, and may include algebraic terms) which also involves time derivatives.

It is known that $\phi_{ss}(x, y, z)$, which has no time dependence, is a solution of the problem under the steady state assumption:¹

$$\mathcal{N}[\phi_{ss}] = 0. \tag{2}$$

All boundary conditions are precisely satisfied by ϕ_{ss} . We now define $\tilde{\phi}$ as some disturbance on ϕ_{ss} :

$$\phi = \phi_{ss} + \epsilon \tilde{\phi} \tag{3}$$

with the understanding that $|\epsilon| \ll 1$. Since both ϕ and ϕ_{ss} are required to satisfy identical boundary conditions, then $\tilde{\phi}$ satisfies “homogeneous” (it is fancy talk meaning zero) boundary conditions. Substituting eq.(3) into

¹We confine our attention here to the stability of steady state solutions. The ideas involved can readily be extended to dealing with the stability of time-periodic solutions.

eq.(1) and linearizing (dropping all terms containing ϵ to a power higher than one), we obtain:

$$\mathcal{L}[\tilde{\phi}] \approx 0 \quad (4)$$

where $\mathcal{L}[\cdot]$ is now a linear operator, containing differential (ODE and/or PDE) operators and also time derivatives.

It is totally obvious that $\tilde{\phi} = 0$ is an exact solution to eq.(4). The issue here is: what if $\tilde{\phi}$ is asked to honor some tiny weenie non-zero initial conditions? (or to handle some tiny weenie non-zero boundary conditions?). Could some tiny weenie non-zero initial disturbance grow and grow to eventually blow everything up? If so, the ϕ_{ss} of interest—which is an exact solution of $\mathcal{N}(\phi) = 0$ —is *linearly unstable*.

3.3 Examples

ODE : Consider

$$\mathcal{N}[\phi] = \frac{d\phi}{dt} + F(\phi) = 0 \quad (5)$$

where $F(\phi)$ is some differentiable (algebraic) function of ϕ . The steady state solution ϕ_{ss} is then simply a root of the algebraic equation $F(\phi_{ss}) = 0$. The linearized equation for $\tilde{\phi}$ is:

$$\mathcal{L}[\tilde{\phi}] = \frac{d\tilde{\phi}}{dt} + J(\phi_{ss})\tilde{\phi} = 0 \quad (6)$$

where $J \equiv \frac{\partial F}{\partial \phi} = J(\phi_{ss})$ is now a known constant. It is clear (since this simple ODE can be solved analytically) that ϕ_{ss} is linearly unstable if $J(\phi_{ss}) < 0$. For example, $F = 1 - \phi^3$; the steady solution $\phi_{ss} = 1$ is linearly unstable. Any non-zero tiny weenie value used for initial condition for $\tilde{\phi}$ will blow up as time marches on.

PDE : Consider

$$\mathcal{N}[\phi] = \frac{d\phi}{dt} + F(\phi) - \mu \nabla^2 \phi = 0 \quad (7)$$

where μ is a positive number. Now the steady state solution is $\phi_{ss}(x, y, z)$. The linearized equation for $\tilde{\phi}$ is:

$$\mathcal{L}[\tilde{\phi}] = \frac{d\tilde{\phi}}{dt} + J(\phi_{ss})\tilde{\phi} - \mu \nabla^2 \tilde{\phi} = 0. \quad (8)$$

To dramatize the illustration, let's say the boundary condition is that $\phi = 1$ on some arbitrary closed boundary surface enclosing a volume of interest. Thus $\phi_{ss} = 1$ is also an exact solution for the special choice of $F = 1 - \phi^3$ even when μ is not zero. We know $\phi_{ss} = 1$ is linearly unstable when $\mu = 0$. How about when μ is some positive number?

The possibility that diffusion can stabilize an otherwise unstable system can be seen as follows. Multiplying eq.(8) by $\tilde{\phi}$, we obtain:

$$\frac{d}{dt} \left(\frac{\tilde{\phi}^2}{2} \right) = -J(\phi_{ss})\tilde{\phi}^2 + \mu \left(\nabla \cdot \tilde{\phi} \nabla \tilde{\phi} - (\nabla \tilde{\phi})^2 \right) \quad (9)$$

where the $\tilde{\phi} \nabla^2 \tilde{\phi}$ term has been massaged a bit. Integrating over the volume of interest, and using the divergence theorem to convert the divergence term to a surface integral which vanishes because $\tilde{\phi}$ is zero on the boundary, we obtain:

$$\frac{d}{dt} \left(\int_{vol} \frac{\tilde{\phi}^2}{2} d\varepsilon \right) = \int_{vol} \left(-J(\phi_{ss})\tilde{\phi}^2 - \mu(\nabla \tilde{\phi})^2 \right) d\varepsilon \quad (10)$$

This equation confirms that when $\mu = 0$, ϕ_{ss} is linearly unstable if $J(\phi_{ss}) < 0$ —the sign of the right hand side is always, thus the magnitude of the (positive) integral on the left hand side will always increase with time without bound. But it is now clear that for “sufficiently” large $\mu > 0$, this conclusion could be reversed—the right hand side could become negative.²

OK. You now want to know what positive, finite value of μ can stabilize the steady state solution? If you need the precise value of this number, then we need to actually “solve” the stability problem.

Eigenvalue problems In general, \mathcal{N} and \mathcal{L} can involve time derivatives of more than first order. For our present exposition, we will limit ourselves to the simplest case—when both \mathcal{N} and \mathcal{L} involve only the first order derivative of time. In other words, we consider the special case:

$$\mathcal{N}[\phi] = \frac{\partial \phi}{\partial t} + \mathcal{F}[\phi] \quad (11)$$

²This is a very popular “trick,” but it does not work all the time. Rayleigh used this trick to deduce the celebrated Rayleigh’s Inflexion-point criterion for inviscid stability—he just needed to respect the fact that the Orr-Sommerfeld equation is complex.

where \mathcal{F} has no time derivatives involvement at all—only spatial derivatives (including zeroth order spatial derivative) are involved. The linearized equation for $\tilde{\phi}$ can now be written as:

$$\mathcal{L}[\tilde{\phi}] = \frac{\partial \tilde{\phi}}{\partial t} + \mathcal{J}_{ss}[\tilde{\phi}]. \quad (12)$$

where the definition of the linear (spatial) differential operator $\mathcal{J}_{ss}[\cdot]$ should be obvious.

It should now be clear how the classical concept of eigenvalues comes in. Consider the eigenvalue problem for the linear differential operator $\mathcal{J}_{ss}[\cdot]$:

$$\mathcal{J}_{ss}[\Phi_n] = \lambda(n)\Phi_n, \text{ (no summation) } \quad n = 1, 2, \dots, \quad (13)$$

where the $\lambda(n)$'s are eigenvalues, and the Φ_n 's are eigenfunctions which are required to satisfy some homogeneous boundary conditions such as being zeros on the geometric boundary of the region. If you do a standard separation of variables on eq.(12), you will wound up with a whole litter of eq.(13)'s staring back at you. Now, what is the physical dimension of $\lambda(n)$? What does it mean?

How does one go about finding eigenfunctions and eigenvalues in the modern computer age?

In your math classes, you have seen this standard eigenvalue problem:

$$\frac{d^2\Phi_n}{dx^2} + \omega^2\Phi_n = \lambda(n)\Phi_n \quad (14)$$

$$\Phi_n(0) = \Phi_n(d) = 0 \quad (15)$$

where ω and d are given constants. We all remember the answer:

$$\lambda_n = \omega^2 - \frac{n^2\pi^2}{d^2}, \quad (16)$$

$$\Phi_n(x) = \sin(n\pi\frac{x}{d}), \quad (17)$$

$$n = 1, \dots \quad (18)$$

This is simple stuff, because we know how to analytically solve constant coefficient ODEs. Now what do you do if ω^2 is some fancy function of

x (and you know for sure that no one, not even Mr. Bessel, had solved this problem previously), and all you have is a computer?

If we accept the proposition that any reasonable spatial function can be expanded as an infinite sum of our eigenfunctions, then the problem of stability is solved once the eigenvalues $\lambda(n)$'s (which may be complex numbers) are found—whenever one or more $\lambda(n)$'s have negative real parts, the steady state solution ϕ_{ss} is linearly unstable (with respect to perturbations in the initial condition, however tiny weenie).

3.4 Finite amplitude stability issues

It is easy to see that there are situations when a given solution is stable when the perturbation is very small, but can become unstable when the perturbation is not small. This is a much more difficult problem—we must include terms of higher order such as ϵ^2 in the analysis. Such problems are usually treated on a case by case basis.

4 Stability of fluid flows

4.1 Some caveats on theories

It is fair to state that the theories of stability of fluid flows are not very pretty. Many *ad hoc* assumptions must be made to render the problem “tractable,” and computations—both the old fashioned kind and the modern supercomputer kind—are difficult.

If the $\mathcal{L}[\cdot]$ linear differential operator had constant coefficients, the problem would be very simple. The special eigenfunctions are simply sines, cosines and exponentials (more elegantly expressed in terms of complex variables), and separation of variables always work. For fluid flow stability studies, the $\mathcal{L}[\cdot]$'s as a rule do not have constant coefficients. So *ad hoc* assumptions are made to make the coefficients as accommodating as possible. The most important *ad hoc* assumption is the so-called “parallel flow” assumption. Suppose one is interested in the stability of the Blasius laminar boundary layer solution. Instead of asking whether the u velocity profile which is a function

of $y/\delta(x)$, it ignores the x -dependence of $\delta(x)$ in the basic streamwise velocity profile, and set the basic vertical velocity profile to zero. Essentially, this assumption says the basic steady flow under study is a parallel flow—which is clearly inconsistent with the original governing partial differential equations of Navier and Stokes. The only rationale for using it is that it is intuitively reasonable, that it enables the mathematics to proceed, and most importantly, it yields many reasonable results in comparison with experiments. It does not have impeccable mathematical justifications.

4.2 Navier-Stokes or Boundary Layer Equations?

Having found a velocity profile from Prandtl's boundary layer equations, what equations would you go to for the study of its stability properties? The suggestion that one would go to Prandtl's boundary layer equations is certain reasonable.

Yet, this is not what is done. The famed Orr-Sommerfeld equation, obtained with the help of the *ad hoc* parallel flow assumption, is derived from the Navier Stokes equations. Interestingly, the final equation was a fourth order ODE for the perturbation v velocity, with a $2 - 2$ split in the (homogeneous) boundary conditions.³

Can you justify the Orr-Sommerfeld use of Cartesian coordinates for studying stability of boundary layers on a curved surface? I will make some additional remarks in class.

5 The classical stability procedures

We shall limit our exposition to two-dimensional disturbances on a two-dimensional (x, y) steady state boundary layer flow solution.

The basic steady state velocity profile will be denoted by $U(y)$ and $V = 0$ —under the parallel flow assumption. The perturbations velocities and pressure are denoted by u, v and p , respectively. Yes, all of them are functions of x, y and t , *including* the perturbation pressure p . (certainly you remember that in boundary layer theory, the static pressure dependence on y was neglected). All three PDEs participates fully, including the y momentum equation. These three linear PDEs do not have constant coefficients—they have

³Note that the real and imaginary components of the Orr-Sommerfeld equation gives TWO equations for the real and imaginary components of v .

coefficients which are independent of x and t but are y -dependent—because the basic steady state velocity profiles under study had been (blatantly) assumed to be parallel flow.

Here comes separation of variables again. The solutions are assumed to be expressible as a sum of entities of the following form:

$$u(x, y, t) = \mathcal{U}(y) \exp(i\alpha(x - ct)) \quad (19)$$

where i is $\sqrt{-1}$, α is a real number and c is a complex number—both dimensional at this point. The physical dimension of α is reciprocal length (wave number), and c is velocity. The function $\mathcal{U}(y)$ —which is to be solved for—is also complex (while y is always real).

Usually, nobody talks about the boundary conditions at the x boundaries (either upstream for downstream) for boundary layer stability problems. Essentially, the physical question being asked is: if the disturbance is decomposed into these sinusoidal traveling waves, what happens to each of them?

The mathematical question is: given value of α (a real number; its reciprocal is the wave length of the traveling wave in question), what is the associated complex value of c such that \mathcal{U} exists? Eigenvalue problem! If the imaginary part of c is positive, the steady velocity profile under study is linearly unstable. The real part of c has a clear physical meaning: it is the “phase” velocity of that traveling wave. The real part of αc is the frequency, and the imaginary part of αc is usually called the growth (or decay) rate.

Let U_e and δ denote characteristic velocity and the thickness of the parallel flow fluid layer. The mathematical result we are looking for is $c(\alpha, \delta, U_e; U(y/\delta)/U_e)$ —the steady velocity profile $U(y)/U_e$ is included behind the semi-colon to indicate that c depends on it “functionally.” Now, what would be the most sensible dimensionless representation of this result? The prevailing representation is:

$$\frac{c}{U_e} = \text{Some function of } (\alpha\delta, Re_\delta; U(y/\delta)/U_e) \quad (20)$$

where $Re_\delta = \frac{U_e\delta}{\nu}$. The results are presented in a graph (for a given $U(y/\delta)$ profile) with contour lines of the real and imaginary components of c , with $\alpha\delta$ and Re_δ as the ordinate and abscissa, respectively.

Note that Re_δ is proportional to $\sqrt{Re_L}$ where L is the characteristic length of the problem. For the Blasius problem (where $L = x$), the proportional factor is approximately 5.

You should read White to learn what are “Rayleigh’s Inflexion-point Theorem” and “Scuries’ Theorem.”

6 What do we learn from the results?

The locus of $c_i = 0$, the imaginary part of c , divides the graph into a stable ($c_i < 0$) and an unstable ($c_i > 0$) region. This locus is called the neutral curve.

The generic picture as described on White’s Fig. 5-7 on page 351 is the following (the displacement thickness δ^* was used instead of the vague δ). For Re_{δ^*} less than a certain critical number, $Re_{\delta^*,crit}$, the flow is stable for all $\alpha\delta^*$. For higher Re_{δ^*} , there is an island of instability, occurring in the region where $\alpha\delta^*$ is a number of order unity. The unstable values of c_i ’s (normalized by U_e) in the island are also order unity numbers. For the Blasius problem, $Re_{\delta^*,crit}$ is approximately 520. The Falkner Skan problems shows that unfavorable pressure gradients depress the value of $Re_{\delta^*,crit}$, while favorable pressure gradient enhance it.⁴ For the stagnation point flow ($m=1$), $Re_{\delta^*,crit}$ is over 10,000. The “most unstable” wave length for the Blasius profile is approximately $18\delta^*$ or 6δ . This wave length increases with favorable pressure gradients, and decreases for unfavorable gradients.

The triumph is: the theory does predict a critical Reynolds Number above which the steady laminar boundary layer solution is linearly unstable. Such unstable solutions are not expected to be observed. There are good experimental confirmation of the predictions.

Item #4 on White’s page 351 says “The maximum temporal growth rate is $c_i/U_o \approx 0.0196$ ” and this is apparently a dimensionless answer. Now, the dimensional growth rate is, according to his eq.(5-12) on page 343, αc_i which can be rewritten as $(\alpha\delta)(c_i/U_o)(U_o/\delta)$. The first two brackets are dimensionless, and the values come from theoretical calculations. The dimensional growth rate is therefore estimated by U_o/δ —something that could have been guessed at by dimensional analysis. So how far downstream do you expect the unstable modes to make a mess of your laminar solution? The answer is: many δ ’s downstream.

⁴It is interesting to note that this result is roughly consistent with the transition from laminar to turbulent flow in round pipes—the transition occurs when the Reynolds Number based on diameter is roughly 2300. And pipe flows have favorable pressure gradients.

7 Homeworks

#1 : Do White's Problem 5-6 on page 392. I have not done this problem personally, but I assume it is a doable problem in the age of Matlab.

Some comments may be helpful to you:

1. Note that the velocity profile White provided for you is supposed to be used only for $0 \leq y \leq \delta$. At $y = \delta$, it gives $u/U = 1$.⁵ So, if you begin at $y = 2\delta$ and integrate inward, you should use $u/U = 1$ between $y = 2\delta$ and $y = \delta$. You should immediately see that Matlab is not needed for the chore here.
2. Remember c is complex—if it were purely real, then things blow up at when it equals U .
3. And of course the unknown v is complex ($v = v_r + iv_i$). And you have to compute both components.
4. Now when the viscous terms are included, we know the Orr-Sommerfeld equation allows us to impose the no-slip condition. Now White wants you to get the inviscid answers by ignoring the viscous terms, he gives you the impression that the the same no-slip boundary conditions can be applied. Do not assume he is right! Think about it, and talk to your fellow students, and watch for my comments in class. What boundary conditions would *you* use for the inviscid Orr-Sommerfeld equation on a non-porous solid wall?
5. After you did the inviscid case, give me your idea on possible strategies on how to handle the viscous case. You are not required to do this—but it should be fun to do some speculations. Hint: perhaps there is a very thin boundary layer near the wall?

#2 : Do White's Problem 5-7.

#3 : Do White's Problem 5-9. Here, you don't have precisely the right data to use. All you have are data for different problems that are generically related to your problem. So you have to speculate a bit to come up with your estimate.

⁵Consistent with notations already in use, White's u/U should be written as U/U_e .