

The Viscous Term and Its Impacts

S. H. (Harvey) Lam

January 10th, 2004
(Edited 2/20/2004)

Abstract

Inviscid fluid mechanics is fluid mechanics without including the viscous term. Here we are studying viscous flows—by definition its goal is to look at the impacts of the *Newtonian* viscous term. The first issue is: how to express the simple Newtonian formula for the viscous stress in a clean, elegant mathematical manner? We shall look at the Navier-Stokes restatement of the Newtonian viscosity. Then we will rederive all the conservation laws—including the Fourier Heat Conduction Law in the energy equation. An emphasis of this section is that one should always state the physical idea of the equation you are trying to write first in English, then write the equation. Conversely, when you see an equation, one should always read it out loud in English, making sure that you got the substance of the message.

1 The Stress Tensor

Given a surface element $d\sigma$ with unit normal \mathbf{n} in a fluid medium, a surface force \mathbf{f} (a vector, with unit of force per unit area) acts on the surface by the presence of the fluid. When the fluid is at rest, the surface force is called *pressure* (usually denoted by 'p'), and its direction is normal to the surface (parallel to \mathbf{n}). Mathematically, we have $\mathbf{f} = -p\mathbf{n}$, which says that the fluid on the side pointed to by \mathbf{n} exerts a normal force (per unit area) on the surface in the direction opposite to \mathbf{n} .

What happens when there is motion? Newton did a thought experiment of two sliding parallel plates a fixed distance apart, and

concluded that for a viscous fluid the surface force \mathbf{f} no longer needs to be normal to the surface. Mathematically, the “stress” (force per unit area) on the surface element $d\sigma$ is eloquently expressed as follows:

$$\mathbf{f} = \mathbf{n} \cdot \bar{\boldsymbol{\tau}} \quad (1)$$

where $\bar{\boldsymbol{\tau}}$ is a *tensor*, and is called a *stress tensor*. In index notation, this is written as:

$$f_j = n_i \tau_{ij} = \sum_{i=1}^3 n_i \tau_{ij}. \quad (2)$$

Note, $\bar{\boldsymbol{\tau}}$ is an array of 9 numbers. Physically, τ_{ij} is the force per unit area in the j -th direction on the surface element (of a tiny cube) whose outward unit normal is in the i -th direction.

In general, we can introduce the matrix $\bar{\boldsymbol{\varepsilon}}$ to represent the viscous contribution to $\bar{\boldsymbol{\tau}}$:

$$\bar{\boldsymbol{\tau}} = -p\bar{\mathbf{I}} + \bar{\boldsymbol{\varepsilon}} = -p\delta_{ij} + \varepsilon_{ij}. \quad (3)$$

where p is pressure (a scalar), δ_{ij} and $\bar{\mathbf{I}}$ are notations for the Kronecker Delta (the latter is usually known as the identity tensor). For a quiescent fluid, $\bar{\boldsymbol{\varepsilon}}$ is zero. It represents the surface stress which can arise by fluid motion.

We shall postpone the “constitutive” relation for $\bar{\boldsymbol{\varepsilon}}$ proposed by Navier and Stokes until later.

2 Warming up with the Continuity Equation

The Law of Conservation of mass, when stated in English, is:

The mass of a material body does not change.

When stated using mathematics, it is:

$$\frac{d}{dt} \int \int \int_{\text{Volume}(t)} \rho dv = 0 \quad (4)$$

where ρ is density (mass per unit volume) and the volume integration is taken on a time-dependent volume which encloses the arbitrarily chosen “material” of interest which is being convected by the velocity

field \mathbf{V} . After a standard song and dance, the following, called the continuity equation, is derived:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \quad (5)$$

Expanding the divergence term, we arrive at an alternative form of the continuity equation:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot (\mathbf{V}) = 0. \quad (6)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \quad (7)$$

which is called the “substantial derivative.” Physically, it is the rate of change with respect to time *following* a tiny moving identified material parcel.

3 The Momentum Equation

The Law of Conservation of momentum (usually called Newton’s Second Law of Motion), when stated in English, is:

The time rate of change of momentum of a material body equals to the net external (surface and body) forces acting on it.

It is understood that momentum (a vector) is mass (a scalar) times velocity (a vector).

When stated using mathematics, it is:

$$\frac{d}{dt} \int \int \int_{\text{Volume}(t)} \rho \mathbf{V} dv = \mathbf{S} + \mathbf{B} \quad (8)$$

where \mathbf{S} denotes the net surface force and \mathbf{B} denotes the net body force acting on the arbitrarily chosen (moving) material volume of interest.

For electrically conducting fluids, \mathbf{B} would include the Lorentz force which takes center stage. For non-conducting fluids, gravity is the most common body force of interest. In general, it would be represented by a volume integral:

$$\mathbf{B} = \int \int \int_{\text{Volume}} \mathbf{b} dv \quad (9)$$

where \mathbf{b} denotes body force per unit volume. For gravity, $\mathbf{b} = \rho \mathbf{g}$ where \mathbf{g} is the gravitational acceleration.

In general, the surface force \mathbf{S} can be written in terms of the surface stress $\bar{\boldsymbol{\tau}}$ already introduced:

$$\mathbf{S} = \int \int_{\text{Surface}} \mathbf{f} d\sigma = \int \int_{\text{Surface}} \mathbf{n} \cdot \bar{\boldsymbol{\tau}} d\sigma \quad (10)$$

Using the Divergence Theorem, we have:

$$\mathbf{S} = \int \int \int_{\text{Volume}} \nabla \cdot \bar{\boldsymbol{\tau}} d\sigma \quad (11)$$

Going through some mandatory song and dance (which we will go through in class), we can readily arrive at the following:

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \nabla \cdot \bar{\boldsymbol{\varepsilon}} + \mathbf{b} \quad (12)$$

If we set $\bar{\boldsymbol{\varepsilon}}$ to zero, we would recover the so-called Euler equation for inviscid fluids.

In ME351B, our focus is on the impacts of the previously neglected $\bar{\boldsymbol{\varepsilon}}$ term when the fluid of interest is a Newtonian fluid.

4 The Energy Equation

The Law of Conservation of energy (usually called the First Law of Thermodynamics), when stated in English, is:

The time rate of change of the sum of internal and kinetic energy of a material body equals to the net external (surface and body) work per unit time \dot{W} done on it plus the net heat transferred per unit time \dot{Q} to it (from the outside world).

It is understood that kinetic energy of a material body is the product of its mass with the square of its velocity divided by two.

Tricky Question: If you were asked to derive the energy equation for a piece of solid (instead of a fluid), would you get kinetic energy involved? Are you happy that kinetic energy is involved?

Another Tricky Question: How come the “potential” energy is not included in the above? Note this presentation is different from the presentation in White.

Everything proceeds uneventfully as in inviscid fluid mechanics, except we need to pay attention to the viscous contributions to \dot{W} and \dot{Q} .

4.1 \dot{Q} vis the Fourier Heat Conduction Law

This is straightforward:

$$\dot{Q} = \int \int_{\text{Surface}} (-\mathbf{n}) \cdot (-\kappa \nabla T) d\sigma = \int \int \int_{\text{Volume}} \nabla \cdot (\kappa \nabla T) dv \quad (13)$$

where κ is the heat conductivity, a material property, of the fluid, and T is the temperature of the fluid.

4.2 \dot{W} due to the Surface Force \mathbf{f}

The work done per unit time per unit surface area by the surface force is $\mathbf{f} \cdot \mathbf{V}$. So we have:

$$\dot{W}_{\text{Surface}} = \int \int_{\text{Surface}} \mathbf{f} \cdot \mathbf{V} d\sigma \quad (14)$$

$$= \int \int_{\text{Surface}} \mathbf{n} \cdot \bar{\boldsymbol{\tau}} \cdot \mathbf{V} d\sigma \quad (15)$$

$$= \int \int_{\text{Surface}} \mathbf{n} \cdot (-p\bar{\mathbf{I}} + \bar{\boldsymbol{\varepsilon}}) \cdot \mathbf{V} d\sigma \quad (16)$$

$$= \int \int \int_{\text{Volume}} \nabla \cdot (-p\mathbf{V} + \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{V}) dv \quad (17)$$

The work done per unit time per unit volume by the body force is $\mathbf{b} \cdot \mathbf{V}$. So we have:

$$\dot{W}_{\text{Volume}} = \int \int \int_{\text{Volume}} \mathbf{b} \cdot \mathbf{V} dv. \quad (18)$$

Putting things together, we have (after the same song and dance):

$$\rho \frac{D}{Dt} \left(e + \frac{V^2}{2} \right) = \nabla \cdot (-p\mathbf{V} + \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{V}) + \mathbf{b} \cdot \mathbf{V} + \nabla \cdot \kappa \nabla T \quad (19)$$

$$= -\nabla \cdot (p\mathbf{V}) + \mathbf{b} \cdot \mathbf{V} + \underbrace{\nabla \cdot (\bar{\boldsymbol{\varepsilon}} \cdot \mathbf{V}) + \nabla \cdot \kappa \nabla T}_{\text{viscous terms}} \quad (20)$$

The last two terms on the right hand side of eq.(20) are the new boys in town.

4.3 Another equation with the same unit

Let us take the inner (dot) product of eq.(12) with Vv . We have:

$$\rho \frac{D(V^2/2)}{Dt} = -\mathbf{V} \cdot \nabla p + (\nabla \cdot \bar{\bar{\epsilon}}) \cdot \mathbf{V} + \mathbf{b} \cdot \mathbf{V}. \quad (21)$$

Note: this is NOT an energy equation, even though it looks like an energy equation, and it has the same dimension as the energy equation. It is a counterfeit (my private terminology) energy equation!

If so desired, we can get rid of the kinetic energy term from our real energy equation using this counterfeit energy equation.

5 The Entropy Equation

For materials in thermodynamic equilibrium, a state variable called *entropy* S exists. This is the famed Second Law of Thermodynamics.

The definition of S is (using standard notations from thermodynamics):

$$dS = \left(\frac{dQ}{T} \right)_{\text{reversible}} \quad (22)$$

Exploiting this definition, the following is called the *differential equation of state* of a gas:

$$dE = TdS - pdV \quad (23)$$

where E is internal energy, T is absolute temperature, p is pressure, and V is volume. In terms of per unit mass entities, this equation becomes:

$$de = Tds - pd\left(\frac{1}{\rho}\right) \quad (24)$$

where e is specific internal energy, s is specific entropy, and ρ is density (mass per unit volume).¹

In a flow field of fluid in thermodynamic equilibrium, we can write:

$$\frac{De}{Dt} = T \frac{Ds}{Dt} - p \frac{D}{Dt} \left(\frac{1}{\rho} \right) = T \frac{Ds}{Dt} - \frac{D}{Dt} \left(\frac{p}{\rho} \right) + \frac{1}{\rho} \frac{Dp}{Dt} \quad (25)$$

or

$$\frac{Dh}{Dt} = T \frac{Ds}{Dt} + \frac{1}{\rho} \frac{Dp}{Dt} \quad (26)$$

¹The spectacular claim of this innocent formula is that the algebraic equation of state the material in question is guaranteed to make the equation for ds a “perfect differential.”

where the specific *enthalpy* h is defined by:

$$h \equiv e + \frac{p}{\rho} \quad (27)$$

You now have everything needed to find the governing equation for Ds/Dt —proposed by Navier and Stokes.

6 The Navier-Stokes $\bar{\bar{\epsilon}}$

Here comes the Newtonian Navier-Stokes $\bar{\bar{\epsilon}}$:

$$\bar{\bar{\epsilon}} = \mu (\nabla \mathbf{V} + (\nabla \mathbf{V})^T) + \lambda (\nabla \cdot \mathbf{V}) \bar{\mathbf{I}} \quad (28)$$

where superscript 'T' indicates 'transposed,' μ is *viscosity*, and λ is something that needs further discussion. Note that $\bar{\bar{\epsilon}}$ is a *symmetric* second order tensor. We shall discuss how to justify this simple and elegant formula in class.

What is pressure? Normally, we understand pressure to be a scalar—it is the normal force acting on a differential surface element, independent of the orientation of the element. Now, with this stress tensor, we find that in general we expect $\epsilon_{11} \neq \epsilon_{22} \neq \epsilon_{33}$. Thus the meaning (and definition) of pressure is now in trouble. Stokes proposed a way out: lets “define” pressure to be the average of any three mutually perpendicular normal forces on a tiny cube. In other words, Stokes proposed that $\epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0$. (this is not science, it is a faith-based hypothesis). Imposing this condition, we find that $\lambda = -2\mu/3$. To avoid the Stokes hypothesis, we can define μ' by:

$$\lambda = -\frac{2\mu}{3} + \mu' \quad (29)$$

and μ' is called the *second coefficient of viscosity*—it is zero if you adopt the Stokes hypothesis. Note that μ' must be positive. See White eq.(2-47) on (p. 72).

In Cartesian coordinates, we have for $\bar{\bar{\epsilon}}$:

$$\epsilon_{ij} = \mu \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) + \lambda \frac{\partial V_k}{\partial x_k} \delta_{ij}. \quad (30)$$

When μ is not a constant, the divergence of $\bar{\bar{\epsilon}}$ contains lots of messy terms. When μ and μ' are constants, the divergence of $\bar{\bar{\epsilon}}$ is given by:

$$\nabla \cdot \bar{\bar{\epsilon}} = \mu \nabla^2 \mathbf{V} + \left(\frac{\mu}{3} + \mu'\right) \nabla(\nabla \cdot \mathbf{V}), \quad \mu, \mu' = \text{constant assumed.} \quad (31)$$

It is clear that μ' only play a role in compressible flows.

7 The No-Slip Condition

For inviscid flows, the boundary condition on a solid surface is that the volume flux to the surface is zero. In other words, nothing flows across the fluid-solid interface.

For viscous flows, the viscous term renders the PDE *elliptic* in mathematical character. Specifically, it demands more information on the solid surface.

The boundary condition for a viscous flow problem is:

There is no slip between the fluid and the solid at the interface.

This is affectionately called the *no-slip* condition by all fluid mechan-icsts.

8 More on Vector and Index Notations

Vector notations look pretty and elegant, while index notations—with the help of Einstein’s Summation Convention—is more explicit. In White, he often mixes both notations in the same equation. This is because when things gets complicated (when tensors are involved), index notation is less confusing.

Suppose you want to take the inner product of \mathbf{A} with $\mathbf{B} \cdot \nabla \mathbf{C}$. In index notation, this is very straightforward:

$$\text{Answer} = A_i B_k \frac{\partial C_i}{\partial x_k}. \quad (32)$$

There are three correct ways of writing this in vector form: $\mathbf{A} \cdot (\mathbf{B} \cdot \nabla \mathbf{C})$ or $(\mathbf{B} \cdot \nabla \mathbf{C}) \cdot \mathbf{A}$ (the parentheses are important!) or $\mathbf{B} \cdot \nabla \mathbf{C} \cdot \mathbf{A}$. The third way is generally understood using the “proximity rule”—the first dot sums the indices of \mathbf{B} and ∇ , and the second dot sums the indices of \mathbf{C} and \mathbf{A} . In other words, when there is uncertainty, *proximity* wins.

Suppose you want to sum *both* indices of the product $\bar{\bar{\mathbf{I}}}$ with $\nabla \mathbf{A}$. This is denoted by the double dot “:” notation:

$$\bar{\bar{\mathbf{I}}} : \nabla \mathbf{A} = \delta_{ij} \frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial x_i} = \nabla \cdot \mathbf{A}. \quad (33)$$

Homework 1: In cylindrical coordinates (r, θ) , work out $\epsilon_{r\theta}$ —the $r\theta$ component of $\nabla \mathbf{V} + (\nabla \mathbf{V})^T$ —and check your answer with bottom of page 575 in White.

Homework 2: What is $\bar{\bar{\mathbf{I}}} : \bar{\bar{\mathbf{I}}}$?

Homework 3: Consider the flow of an incompressible liquid (ρ is a constant). How many unknowns do we have to fully describe (and to be solved for) for the flow field? How many equations (and what are they) do you have for them? Consider now the flow of a gas (such as air). How many unknown now, and how many equations? If you are one equation short, where do you go to find it? What assumption do you need?

Homework 4: Find the equation for Ds/Dt and use the index notation (in Cartesian coordinates)—starting from the given continuity, momentum and energy equations. You can look at White. But you must be able to do it independently. What happens if the flow is “inviscid”?

Homework 5: The *vorticity* vector $\bar{\bar{\boldsymbol{\omega}}}$ is defined by:

$$\bar{\bar{\boldsymbol{\omega}}} \equiv \nabla \times \mathbf{V}. \quad (34)$$

Take the curl of the momentum equation (assume μ and λ to be constants to simplify the messy algebra; but ρ is not a constant) and find an equation for $D\bar{\bar{\boldsymbol{\omega}}}/Dt$. Look at the equation. Can you hear it say something to you?

Homework 6: Let u denote the velocity parallel to a local solid surface element, and let y denote the vertical coordinate. For a solid at rest, the no-slip boundary condition is $u(y=0)=0$. Now suppose someone comes along and says he thinks $u(y=0)$ ought to depend on the value of $\partial u/\partial y$. What would a person well educated in the art of dimensional analysis say?

Homework 7: Suppose you are in a rocket spaceship which is travelling in a perfectly straight trajectory with non-constant velocity $V_o(t)$. You are an astronaut in the spaceship, and you want to

write down the fluid mechanics conservation equations in your local (moving with the spaceship) coordinates. So, you would represent the moving coordinate effect by an apparent “gravity” $\mathbf{g}(t)$. Discuss the energy equation—after looking at the approach taken by White.