Overview of Turbulenct Flows

Harvey S. H. Lam

February 29, 2004

Abstract

Turbulence is a big subject with a huge literature. We can only attempt to provide an overview here. The bottom line is: an educated fluid mechanist must be able to make respectable engineering estimates—in practical situations.

1 Introduction

It is very important to keep in mind some visual images of turbulence. It is obvious that turbulent flows do not look like laminar flows. The distinquishing characteristics is that the flow is unsteady, and theoretical musings based on laminar steady flows concepts are totally wrong—even when attention is confined to problems that can be adequately described by time-averaged entities.

To clear the air, I believe it is generally agreed (in 2004) that turbulence is stil an unsolved problem. Starting with the premise that the unsteady Navier-Stokes equations are the governing equations for turbulent flows, the challege is how to get a self-consistent theory which can describe the coarse-grain features of the unsteady flow field without being bothered (or being stucked) by the fine-grain structures. It is my personal opinion that it is an unsolvable problem in the sense that there exists no "first principle" formulation which can yield a self-contained coarse-grain theory. All turbulence theories needs some speculative assumptions on the fine-grain structures, and the quality of a good theory depends on how sensible the speculations are, and how good the predicted answers are for real world problems.

We shall limit our discussion to constant property two-dimensional flows (ρ and μ are constants).

2 The experimental database

How badly off are the laminar predictions for turbulent flow?

The answer is: absolutely terrible.

pipe flows: The laminar theory says the velocity profile is always a parabola, and the "friction factor" (C_f) is given by $64/\sqrt{Re_D}$. See White's Fig. 6-13 on page 423. For Re_D above $2000 - 4000$, Blasius recommends Eq.(6-51) on page 422—for smooth pipes only. Note that the turbulent C_f decreases very weakly with increasing Re_D. Look at Eq.(6-54), Prandtl's recommendation. Note that C_f appears on both sides of the equation! (Remember: log is a very weak function of its argument). Roughness data is given in §6-5.4 on page 426. The recommended correlation is Eq.(6-66) on page 428.

flat plate: The laminar Blasius formula for C_f is $0.664/\sqrt{Re_x}$. A curve-fit of the data is given by Eq.(6-70) on page 430, which says for turbulent boundary layers we have $C_f = 0.027/Re_x^{1/7}$. Now, a 1/7 power is a very weak dependence. When your Reynolds number goes up 10 million fold, your new C_f goes down by a factor of 10. Most interestingly, the thickness of a flat plate turbulent boundary layer δ is, according to Eq.(6-70), $(0.16x)/Re_x^{1/7}$ (the laminar thickness is $(5.0x)/Re_x^{1/2}$). Is a turbulent boundary thin? Yes. Does it get thinner when the Reynolds number gets bigger? Yes, but very weakly. But in general it is much thicker than its laminar counterparts.

velocity profiles: Compare White's Fig.4-11 on page 245 (and Fig.4-6 on page 237) with Fig.6-8 on page 411. In particular, look at Fig.6-8 and remember that the no-slip condition is not to be ignored in turbulent flow.

3 The turbulent stress term

We confine our attention to problems that make sense when we consider a time-averaged description.

We represent all flow field dependent variables by a barred symbol and a primed symbel, representing the time-averaged and the fluctuation time-dependent components. It is totally straightforward to show, as White did in the book, that time-averaged momentum equation now has an extra term—representing the transport of momentum across the surface of a control volumne by the fine-grain, fluctuating turbulent motions. This new term is called the turbulent stress term. In index notation, we have:

$$\tau_{ij}^{turb} = -\overline{\rho u_i' u_j'}. \tag{1}$$

As you can see, the turbulent stress tensor needs detailed information on the "correlation" of the relevant velocity fluctuations.

Obviously, we must also impose the condition that τ_{ij}^{turb} must be zero for uniform flows and solid body rotations, etc. One obvious candidate for modeling this stress tensor is:

$$\tau_{ij}^{turb} = \mu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right). \tag{2}$$

The new twist is that the turbulent viscosity (denoted by μ_t by White and affectionately called "eddy viscosity" in the community), is no longer expected to be a material property. It is expected to be a property of the turbulence. The challenge is to figure out how to model it.

What is the physical dimension of μ_t? It is the product of density, velocity and a length. There is no question what density is relevant. What velocity comes to mind? What length comes to mind?

3.1 The concept of a "filtered" solution

Conceptually, given a messy time dependent solution, there is more than one way to decompose it into the barred and the primed entities. Instead of saying the barred entities are the "long time" averaged entities which are themselves time-independent, we can say they are averaged over some specified averaging time period. Or, we can say the averaging procedure to obtain the barred entities is done over some finite time interval and some finite space volume. Whatever we do, the mathematics will tell us that the governing PDEs for the barred entities will have the same extra terms—generically. Often, the barred entities are called the filtered solutions.

What is going on physically? Essentially, the barred entities is being influence by momentum and energy fluxes transported across surfaces of control volumes by the high frequency, short wave length part of the full (unfiltered) solution. Essentially, this is the same physical process that gives rise to the fluid surface stress tensor and the Fourier heat conduction law—molecular random motions were responsible.

4 Global μ_t?

Let's consider the classical problem of flows in a circular straight pipe. Let the centerline (averaged) velocity be U_*, the diameter be D_* and the (constant) density and (molecular) viscosity be ρ_* and μ_*. The laminar friction factor $\mathcal{F}$ is simply $64/Re_D$.

What kind of theory do I get if I replace the molecular viscosity by a constant turbulent eddy viscosity μ_t which is a flow property?

Doing the chores of dimensional analysis, we conclude that there are two independent dimensionless parameters. Let's pick the following:

$$\Pi_1 = \frac{\mu_t}{\rho_* U_* D_*}, \tag{3}$$

$$\Pi_2 = Re_D \equiv \frac{\rho_* U_* D_*}{\mu_*}. \tag{4}$$

These two dimensionless parameters are related. In other words, we are looking for:

$$\mu_t = \rho_* U_* D_* \Pi_1(Re_D) \tag{5}$$

where $\Pi_1(Re_D)$ means Π_1 as a function of Re_D. Physically, we expect Π_1 to be a small (compared to unity) positive number.

Since μ_t is by assumption a constant, the new solution of the problem can readily simply obtained by replacing μ_* with $\mu_* + \mu_t$. In other words, we have for the turrbulent friction factor $\mathcal{F}_t$::

$$\mathcal{F}_t = \frac{64}{Re_D}\frac{\mu_t}{\mu_*} = \frac{64}{Re_D}\left(1 + Re_D \Pi_1(Re_D)\right) \approx \underbrace{64\Pi_1(Re_D)}_{R_D>>1} \tag{6}$$

If we peek at the Moody Diagram, we see that for rough pipes, Π_1 tends to a small constant for very large Re_D. For smooth pipes, Π_1 decreases slowly with increasing Re_D. So Π_1 also has a dependence on wall roughness. In any case, $\Pi(Re_D, \text{roughness})$ can be determined empirically (and confirm that Π_1 is always a small number compared to unity). However well that job is done, it is understood that the velocity profiles of this "theory" remain a parabola, and are no good at all. In other words, this is not a respectable "theory" (I cooked it up last night).

4.1 Similar solutions

plane jet: See White's Eq.(6-146) and (6-147, 149). You need to be given U_o, x_o and b_o in order to proceed. The big deal is that (see top of page 474) a turbulent plane jet grows at a half-angle of 13°, independent of the Reynolds Number!

others: Circular jet, plane mixing layer, and turbulent wakes expositions by White (page 474-481) all gave Reynolds Number independent answers. This conclusion is NOT a surprise, since all assumed a turbulent viscosity which was independent of molecular viscosity. They are useful results because they had experimental validation (for the Reynolds Numbers considered).

5 Situation near a wall

Physically, laminar viscosity accounts for the momentum exchange between two layers of fluid (moving relative to each other) by molecules jumping across in both directions. Essentially the same physics needs to be accounted for when we have turbulent eddies jumping around.

How far can a turbulent eddy a distance y from a solid wall jump, on the average? Let's call this length "mixing length," ℓ, after Prandtl. Certainly ℓ can't be something bigger than y—remember: there is a wall there. So if we write:

$$\ell = ky \tag{7}$$

we know k is a positive dimensionless number less than unity (to be determined empirically). Now, when this turbulent eddy jumps, what is the velocity difference between the two time-averaged layers of fluid? A reasonable model is:

$$\Delta\bar{u} = \ell \left|\frac{\partial \bar{u}}{\partial y}\right| \tag{8}$$

With this much "physical thinking," we are ready to propose a model for the turbulent viscosity μ_t:

$$\mu_t = \rho(\Delta\bar{u})\ell = \rho k^2 y^2 \left|\frac{\partial \bar{u}}{\partial y}\right|. \tag{9}$$

Remember, k is to be determined empirically. The turbulent shear stress in a turbulent boundary layer is then:

$$\tau_{ij}^{turb} = \rho k^2 y^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \frac{\partial \bar{u}}{\partial y}. \tag{10}$$

Now, if we make the assumption that τ_{ij}^{turb} is roughly constant near a solid wall (and is much larger than μ_* outside of a very thin "laminar sublayer" immediately adjacent to the wall), this equation can be integrated analytically and allows us to determine a velocity profile:

$$\frac{\bar{u}}{\bar{u}_*} = \frac{1}{k} \ln y + \text{constant}. \tag{11}$$

where $\bar{u}_*$ is short-hand notation to represent the constant τ_{ij}^{turb} in the wall region:

$$\bar{u}_* \equiv \sqrt{\frac{\tau_{ij}^{turb}}{\rho}}. \tag{12}$$

Now eq.(11) is not yet pretty—taking the log of a dimensional number is not good form, and there is a integration constant hanging around. Obviously, we got to replace the dimensional y by something dimensionless. What do you think? How about this (what else?):

$$y^+ \equiv \frac{\rho \bar{u}_* y}{\mu_*}. \tag{13}$$

Indeed this is the accepted choice. So eq.(11) becomes:

$$u^+ = \frac{1}{k} \ln y^+ + B, \quad (u^+ \equiv \frac{\bar{u}}{\bar{u}_*}) \tag{14}$$

where B is a dimensionless constant. In the laminar sublayer, the laminar velocity profile must be approximately:

$$\frac{\bar{u}}{\bar{u}_*} = y^+. \tag{15}$$

So, the velocity should follow eq.(15) starting at the wall, and switches over to eq.(14) at some finite value of y^+. You will find graphs in White (fig. 6-11 on page 416) showing experimental data, plotting $\bar{u}/\bar{u}_*$ versus y^+ in semi-log plots.

The constants $k \approx 0.41$ and $B \approx 5.0$ (for a smooth wall) are thus readily determined empirically (see White, page 415). For rough wall, B can be given the job of absorbing the effect of roughness. See White's §6-6.4 on page 433).

6 Momentum Integral Studies

A sensible proposition by Coles to represent a general velocity profile is given by Eq.(6-47) on page 419 of White:

$$u^+ = \frac{1}{k} \ln y^+ + B + \frac{2\Pi}{k} f(\frac{y}{\delta}). \tag{16}$$

where Π is called the Cole Wake Parameter. This velocity profile is supposed to represent both the near wall region (outside of the laminar sublayer) and the region not so close to the wall. The function $f(y/\delta)$ describes the velocity profile there (see White's Eq.(6-46) on page 417; note $f(0) = 0$ and $f(1) = 1$), and Π is a dimensionless parameter, which obviously should depend on the prevailing pressure gradient (see White's Eq.(6-121) on page 451, and the definition of β ($\equiv (\delta_*/(\rho\bar{u}_*^2))(dp_e/dx)$) on White's Eq.(6-42) on page 416. See also $\Pi(\beta)$ as displayed as correlation of available experimental data in fig.(6-27) on page 452).

Another way of writing eq.(16) is:

$$u^+ = \frac{1}{k}\ln\frac{y}{\delta} + (B + \ln\frac{\rho\bar{u}_*\delta}{\mu_*}) + \frac{2\Pi(\beta)}{k}f(\frac{y}{\delta}). \tag{17}$$

This velocity profile contains lots of valuable information. Its credibility comes from the quality of its experimental data correlation.

Most importantly, note that by evaluating eq.(17) at $y = \delta$ where $u^+ = U_e/\bar{u}_*$, the resulting formula gives us (almost) the turbulent friction factor.

The momentum integral equation (White's Eq.(6-28) on page 409) is:

$$\left(\frac{\bar{u}_*}{U_e}\right)^2 = \frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e}\frac{dU_e}{dx} \tag{18}$$

Using eq.(17), we can straightforwardly compute the displacement thickness δ_*, momentum thickness θ in terms of Π, k and the wall friction represented by $\bar{u}_*/U_e$ (see White's Eq.(6-48,49) on page 420):

$$\frac{\delta_*}{\delta} = \frac{1+\Pi}{k}\frac{\bar{u}_*}{U_e}, \tag{19}$$

$$\frac{\theta}{\delta} = \frac{\delta_*}{\delta} - \frac{2 + 3.2\Pi + 1.5\Pi^2}{k^2}\left(\frac{\bar{u}_*}{U_e}\right)^2, \tag{20}$$

$$\frac{U_e}{\bar{u}_*} = \frac{1}{k}\ln(\frac{\rho\bar{u}_*\delta}{\mu_*}) + B + \frac{2\Pi}{k}. \tag{21}$$

No physics is involved in getting these three equations. The above four equations are to be supplemented by

$$\Pi = \Pi(\beta) \tag{22}$$

which is also a correlation (see fig.(6-27) on page 452), where (dimensionless) β is defined by:

$$\beta \equiv \frac{\delta_*}{2\bar{u}_*^2}\frac{dU_e^2}{dx}. \tag{23}$$

Counting the number of unknowns $(\bar{u}_*, \delta_*, \theta, \delta, \Pi, \beta)$ and equations, we are in good shape—even though things are a bit messy. Note that ρ, μ_*, B and k are known constants, and $U_e(x)$ is assumed given. Now, when this system of equations are solved, the problem is solved. The credibility of the empirical theory depends completely on the credibility of the velocity profiles used in its development.

7 The Smagorinsky turbulence model

Dr. Joseph Smargorinsky works with John von Neumann more than 50 years ago. They had an idea: why not solve the equations of fluid mechanics of the whole global atmosphere and predict the weather? Thus von Neumann started on the idea of an electronic computer, built a machine that is now on exhibit in the Smithsonian (total random access memory of 1k), and helped to usher in the digital computer age.

The Smagorinsky model for "global circulation" ran in the early 1960s on 16K of ram. The famous Smagorinsky turbulent kinematic viscosity model is the following:

$$\nu_t = \ell^2 \sqrt{\frac{\partial \bar{u}_i}{\partial \bar{x}_j} \left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i} \right)} \tag{24}$$

where the mixing length is proportional to the grid size used in the computation (near the wall, $\ell = ky$ is used). Variants of this model is still highly respected today. The important point to note is: on a computer, the turbulent viscosity is grid size dependent!

8 The K-ϵ Model

Physically, the turbulent viscosity is the consequence of random transport due to the high-frequency, short wave length unsteady motions. If we denote the kinetic energy content of the eddies by K, then obviously ν_t must depend on K. But dimensionally, ν_t must depend on K and something else. What could it be?

How do the small eddies get their kinetic energies? It is reasonable to say that big eddies break up into smaller eddies, small eddies becomes even small eddies, and eventually the eddies are small enough that molecular viscosity takes over to convert the kinetic energy into heat. So, this suggests that perhaps ν_t should depend on ϵ, the dissipation rate of the kinetic energy. From dimensional analysis, we can immediately conclude:

$$\nu_t = C_\mu \frac{K^2}{\epsilon}. \tag{25}$$

So (C_μ is empirically recommended to be 0.09 in White's Eq.(6-107) on page445), we now also need to commit ourselves to find equations for K and ϵ.

This is quite a popular approach, and has had considerable successes. PDEs are developed for K and ϵ, and additional speculative modeling of terms in these equations are needed (see White's Eq.(6-105a,b) on page 445; mor empirical constants!). White's description is not very extensive or complete (boundary and initial conditions?), and I myself am not fully informed on the latest developments.

9 Turbulent heat transfer

What happens to heat conductivity κ in a turbulent flow? It becomes κ_t, of course. What happens to the Prandtl Number? It becomes Pr_t (and then often the subscript 't' is omitted). The main point is that Pr_t is not a material property, but a turbulence property, and that it is $O(1)$. Look at White's Eq.(6-168) and compare it with Eq.(4-80), both equations are shown on page 485. This is the empirical data correlation to substantiate the concept of Reynolds Analogy for heat transfer. Make sure you know what a Stanton Number is (It is defined in White's Eq.(1-98) on page 49).

So since turbulent wall friction is much higher then laminar values, turbulent heat transfer is much stronger than laminar values.

10 My personal view

Unlike laminar viscous flow which has a self-consistent foundation, turbulent flow has a fundamental "closure" problem. It is my opinion that all turbulent theories need empirical support—somewhere in there a speculative assertion had to be made, and solid experimental evidence must be provided to make the assertion credible. Time and again, dimensional analysis plays a significant role.

11 Homeworks

1. Let the unsteady components (actually, the components that are filtered out) of a turbulent flow field be Fourier-decomposed into sines and cosines. In mathematical jargon we are going into "spectral" space (from physical space). The usual notation for the "wave number" of the sines and cosines is k, which has the physical dimensional of reciprocal length.

   It is believed that the dynamics of these unsteady eddies is mainly influenced by the local energy dissipation ϵ (of the filtered solution), which has the physical dimension of velocity square per unit time. This speculation provides the link between the filtered solution and the stuffs being filtered out.

   a: We are interested in $E(k)$, where E is the kinetic energy per unit wave length—it physical dimension is length^3 time^{-2}. The answer that you get is generally known as the Kolmogorov Energy Spectrum—which has good experimental support provided the characteristic Reynolds Number is large.

   b: We are trying to do numerical simulation of a turbulent flow, using a physical grid of size Δ. In other words, the grid used is filtering out any Fourier components with $k > k_{max} \equiv 2\pi/\Delta$. Use dimensional analysis to find out how the turbulent kinematic viscosity ν_t depends on ϵ and Δ. What you find is essentially the Smagorinsky turbulence model.

   c: We want to have an estimate of the total amount of kinetic energy $K(\Delta)$ in all the filtered out eddies (those with $k \geq k_{max}$). Take a crack at it.

2. Given a (semi-finite) smooth flat plate, one meter long in a (constant property flow) of one meter per second. The fluid is air.

   a: Compare the boundary layer thickness (the vague δ) at the end of the plate under the assumption of laminar and turbulent flows starting from $x = 0$ (you can put something near the leading edge to "trip" a laminar boundary layer into turbulence).

   b: Obviously, near the leading edge, the flow is laminar. Where do you expect transition to occur?

   c: Compare the wall frictions of the whole plate under the same assumption as in item a above.

d: Compare the heat transfer capabilities under the same assumption as in item a above.

e: Now, you double the flow velocity. What happens to all your answers?

3. You are working on a problem with Reynolds Number in the range which were relevant to the data shown on Fig.6-29 on page 459 (no pressure gradient).[1] You want to make an estimate on how much adverse pressure gradient is needed to cause flow separation. Take a look at White's Eq.(6-28) on page 450—the Karman's momentum integral equation. Can you make an educated guess?

[1]Note that no Reynolds Number is given by White for these data.