Review of Vector Calculus

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Abstract

This review is simply a summary of the major concepts, definitions, theorems and identities that we will be using in ME351B.

1 Fundamental Definitions

We consider only three dimensional Euclidean space.

A vector is denoted by a bold face symbol, such as $\mathbf{V}$. A Cartesian coordinate system (x, y, z) has three mutually perpendicular straight axes (ordered in accordance to the *right-hand rule*). The Cartesian components of $\mathbf{V}$ are denoted V_x, V_y and V_z. The directions of the three axes are given by their *unit* vectors denoted by $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, respectively.

Magnitude of V is a scalar : (Pythagorean Theorem)

$$|\mathbf{V}| \equiv \sqrt{V_x^2 + V_y^2 + V_z^2} \tag{1}$$

Addition of V and U yields a vector :

$$\mathbf{V} + \mathbf{U} \equiv \mathbf{e}_x(V_x + U_x) + \mathbf{e}_y(V_y + U_y) + \mathbf{e}_z(V_z + U_z) = \mathbf{U} + \mathbf{V} \tag{2}$$

Scalar (or dot, or inner) product of V and U is a scalar :

$$\mathbf{V} \cdot \mathbf{U} \equiv |\mathbf{V}||\mathbf{U}| \cos \Theta = V_x U_x + V_y U_y + V_z U_z = \mathbf{U} \cdot \mathbf{V} \tag{3}$$

where Θ is the angle 'between' the two vectors involved.

Vector (or cross) product of V and U is a vector :

$$\mathbf{V}\times\mathbf{U} \equiv \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ V_x & V_y & V_z \\ U_x & U_y & U_z \end{bmatrix} = -\mathbf{U}\times\mathbf{V},\ |\mathbf{V}\times\mathbf{U}| = |\mathbf{V}||\mathbf{U}||\sin\Theta|. \tag{4}$$

The direction of the vector $\mathbf{V} \times \mathbf{U}$ is pointed to by your right thumb if you first point your four fingers (of your right hand) in the $\mathbf{V}$ direction, then bend them in the $\mathbf{U}$ direction. For example, $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$. This is called the *right-hand rule.*

The product UV is an example of a *tensor.* Note $\mathbf{UV} \neq \mathbf{VU}$.

Gradient of a scalar field is a vector field :

$$\nabla\phi \equiv \lim_{\Delta\upsilon\to 0} \frac{1}{\Delta\upsilon} \int\int_{\text{Surface}} \mathbf{n}\phi \; d\sigma \tag{5}$$

Divergence of a vector field is a scalar field :

$$\nabla\cdot\mathbf{V} \equiv \lim_{\Delta\upsilon\to 0} \frac{1}{\Delta\upsilon} \int\int_{\text{Surface}} \mathbf{n}\cdot\mathbf{V} \; d\sigma \tag{6}$$

Curl of a vector field is a vector field :

$$\nabla\times\mathbf{V} \equiv \lim_{\Delta\upsilon\to 0} \frac{1}{\Delta\upsilon} \int\int_{\text{Surface}} \mathbf{n}\times\mathbf{V} \; d\sigma \tag{7}$$

In the abvoe, $d\sigma$ is a surface element of the *closed* volume (a volume that has an 'inside' and an 'outside') $\Delta\upsilon$, and $\mathbf{n}$ is its *unit outward normal.* **Surely you noticed the analogous roles played by the operator ∇ and the unit outward normal n in the above definitions**. Note: The right hand sides of all the above three equations involves well understood mathematical operations without making any reference to the coordinate system. Thus these are *definitions* of the *operators* on the left hand side when we take the limit of a vanishingly small differential volume element.

Note that $\nabla\mathbf{V}$ was *not* defined by the above. It is meaningful and is a *tensor.* Be careful when you deal with curvilinear coordinates!!!

2 Major Theorems

The following are major theorems that provide relations between volume and surface integrals and surface and line integrals involving the above operators. It is essential that you notice that these theorems are stated without reference to a coordinate system. In other words, they provide the foundation for the *derivation* of the three operators in any coordinate system of interest.

2.1 Divergence Theorems—trivially proved from definitions

All three theorems below are often called *Divergence Theorems*. However, most people would identify only the one involving the divergence operator as the Divergence Theorem.

Gradient :

$$\int\int\int_{\text{Volume}} \nabla\phi \; dv = \int\int_{\text{Surface}} \mathbf{n}\phi \; d\sigma \tag{8}$$

Divergence :

$$\int\int\int_{\text{Volume}} \nabla\cdot\mathbf{V} \; dv = \int\int_{\text{Surface}} \mathbf{n}\cdot\mathbf{V} \; d\sigma \tag{9}$$

Curl :

$$\int\int\int_{\text{Volume}} \nabla\times\mathbf{V} \; dv = \int\int_{\text{Surface}} \mathbf{n}\times\mathbf{V} \; d\sigma \tag{10}$$

2.2 Stokes Theorem

This theorem can be proved from the above defintions:

$$\int\int_{\text{Surface}} (\nabla\times\mathbf{V})\cdot\mathbf{n} \; d\sigma = \int_{\text{Perimeter}} \mathbf{V}\cdot \; d\,\boldsymbol{\ell} \tag{11}$$

where $d\,\boldsymbol{\ell}$ is a differential vector element along the perimeter of the surface in the direction of your four right fingers when your right thumb is pointing in the $\mathbf{n}$ direction. The 'surface' here is not the surface of a closed volume; it is an arbitrary surface with a perimeter as its 'edge.' The proof is straightforward, but somewhat tedius.

3 Useful Vector Identities

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{12}$$

$$\nabla \times (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{C} + \mathbf{B}(\nabla \cdot \mathbf{C}) - \mathbf{C}(\nabla \cdot \mathbf{B}) \tag{13}$$

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \nabla \frac{A^2}{2} - \mathbf{A} \cdot \nabla \mathbf{A} \quad (\textbf{Be careful!!!!}) \tag{14}$$

$$\nabla \times (\nabla \times \mathbf{C}) = \nabla(\nabla \cdot \mathbf{C}) - \nabla^2 \mathbf{C}, \quad (\textbf{Be careful!!!!}) \tag{15}$$

Note that eq.(14) involves the gradient of a vecor—something that is *undefined* so far (so, be careful!). The Laplacian operator ∇^2 is shorthand for $\nabla \cdot \nabla$. Note that eq.(14) and eq.(15) can be used to get rid of $\mathbf{A} \cdot \nabla \mathbf{A}$ and $\nabla^2 \mathbf{A}$ in curvilinear coordinates.

4 Coordinate System

In addition to Cartesian coordinates, we shall need to be competent in dealing with curvilinear coordinates.

4.1 Cartesian and the index notation system

Instead of (V_x, V_y, V_z), it is often convenient to use the *index notation* (V_1, V_2, V_3), to denote the three Cartesian coordinates. The bold-faced $\mathbf{V}$ is then denoted by V_i, with $i = 1, 2, 3$.

As a convenient shorthand, the so-called *Einstein Summation Convention* is usually adopted: Whenever an index is repeated (once) in an expression, the sum of that index (from 1 to 3) is assumed and need not be written out.

We have:

$$\nabla \phi = \frac{\partial \phi}{\partial x_i}. \tag{16}$$

$$\mathbf{V} \cdot \mathbf{U} = \sum_{i=1}^{3} V_i U_i = V_i U_i. \tag{17}$$

$$\nabla \cdot \mathbf{V} = \sum_{i=1}^{3} \frac{\partial V_i}{\partial x_i} = \frac{\partial V_i}{\partial x_i}. \tag{18}$$

To deal with the vector cross product and the curl operator, we need the *permutation tensor* ϵ_{ijk} which is defined by the following properties:

1. $\epsilon_{ijk} = 0$ if $i = j$, or $j = k$, or $k = i$, or $i = j = k$.
2. $\epsilon_{ijk} = 1$ if i, j, k are arranged clockwise.
3. $\epsilon_{ijk} = -1$ if i, j, k are arranged counterclockwise.

With the help of this permutation tensor, we have:

$$\mathbf{V} \times \mathbf{U} = \epsilon_{ijk} V_j U_k. \tag{19}$$

Note that $\epsilon_{ijk} = -\epsilon_{jik} = \epsilon_{jki}$—switching adjacent indices changes sign. The curl can be written as:

$$\nabla \times \mathbf{V} = \epsilon_{ijk} \frac{\partial V_k}{\partial x_j}. \tag{20}$$

4.1.1 The major identity

The Kronecker Delta δ_{ij} (also denoted by $\bar{\bar{\mathbf{I}}}$) is defined by:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{21}$$

This is a most useful identity:

$$\epsilon_{ijk}\ \epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{22}$$

The above are 81 distinct relations. Note that the repeated 'i' index on the left hand side is being summed.

4.2 Curvilinear Orthogonal Coordinate Systems

Let ξ_i (i=1,2,3) denote a three-dimensional curvilinear locally orthogonal coordinate system. The ξ_i's need not have the same physical units. For example, in cylindrical polar (r, θ, z) coordinate, ξ_1 and ξ_3 have the unit of length, while ξ_2 have the unit of radians. Thus, the list of three numbers (ξ_1, ξ_2, ξ_3) may not be a vector in the Euclidean sense: $\xi_1^2 + \xi_2^2 + \xi_3^2$ may or may not make sense physically.

The length (magnitude) of a differential line element $d\,\boldsymbol{\ell}$ in Euclidean Space is given by:

$$(d\ell)^2 = (h_1 d\xi_1)^2 + (h_2 d\xi_2)^2 + (h_2 d\xi_2)^2. \tag{23}$$

where $h_1(\xi_1, \xi_2, \xi_3), h_2(\xi_1, \xi_2, \xi_3), h_3(\xi_1, \xi_2, \xi_3)$ are known "scale factors." For Cartesian coordinates, we have $h_1 = h_2 = h_3 = 1$. For cylindrical polar coordinates (r, θ, z), we have $h_1 = h_3 = 1$ and $h_2 = r$.

Using the definitions of gradient, divergence and curl, we can work out[1] the formulas in curvilinear orthogonal coordinate systems.

[1]Draw a small volume using the chosen coordinate system, and work out the surface integrals carefully, then take the limit as the small volume shrinks toward zero.

Gradient of a scalar :

$$\nabla\phi = \mathbf{e}_1\frac{\partial\phi}{h_1\partial\xi_1} + \mathbf{e}_2\frac{\partial\phi}{h_2\partial\xi_2} + \mathbf{e}_3\frac{\partial\phi}{h_3\partial\xi_3} \tag{24}$$

Divergence of a vector :

$$\nabla\cdot\mathbf{V} = \frac{1}{h_1h_2h_3}\left(\frac{\partial h_2h_3V_1}{\partial\xi_1} + \frac{\partial h_3h_1V_2}{\partial\xi_2} + \frac{\partial h_1h_2V_3}{\partial\xi_3}\right) \tag{25}$$

Curl of a vector :

$$\nabla\times\mathbf{V} = \frac{1}{h_1h_2h_3}\begin{bmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{bmatrix} \tag{26}$$

You can prove the above formula starting from eq.(8)-eq.(11), taking care to account for the fact that the unit outward normal **n** can depend on ξ_1, ξ_2, ξ_3.

5 The Concept of Flux

5.1 Volume Flux by Convection

Let Q, a dimensional scalar, denote the volume flux passing through a specified (user-chosen) surface. The unit of Q is *volume per unit time*, e.g. cubic feet per second, or cubic meter per day, ..., etc.

Mathematically, we have:

$$Q = \int\int_{\text{Surface}}(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{27}$$

where $d\sigma$ is a differential surface element on the surface of interest, **n** is its unit normal (pointing in the direction you choose), and **V** is the velocity of the moving medium. The dot product $\mathbf{n}\cdot\mathbf{V}$ is the component of **V** in the direction of **n** at the surface element $d\sigma$.

You can check that the units on both sides agree.

The *differential volume flux* convected through the surface element $d\sigma$, denoted by dQ, is given by:

$$dQ = (\mathbf{n}\cdot\mathbf{V})d\sigma. \tag{28}$$

Make sure you feel comfortable with this.

5.2 Other fluxes by convection

Other fluxes follow naturally: Let ρ denote mass density (mass per unit volume):

$$\text{Mass flux} = \int\int_{\text{Surface}} \rho(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{29}$$

Let $\rho\mathbf{V}$ denote momentum density (momentum per unit volume):

$$\text{Momentum flux} = \int\int_{\text{Surface}} \rho\mathbf{V}(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{30}$$

Let e denote internal energy density (energy per unit volume):

$$\text{Thermal energy flux} = \int\int_{\text{Surface}} \rho e(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{31}$$

Let K denote kinetic energy density (kinetic energy per unit volume):

$$\text{Kinetic energy flux} = \int\int_{\text{Surface}} \rho\frac{\mathbf{V}\cdot\mathbf{V}}{2}(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{32}$$

Let W denote pollutant density (mass of pollutant per unit volume):

$$\text{Pollutant flux} = \int\int_{\text{Surface}} W(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{33}$$

Let G denote gold fish density (number of gold fish per unit volume):

$$\text{Gold fish flux} = \int\int_{\text{Surface}} G(\mathbf{n}\cdot\mathbf{V})d\sigma \tag{34}$$

...

6 General Advices and Remarks

With these few pages of notes, you can handle any curvilinear coordinates problems. The most important thing to remember is: **the gradient of a vector in curvilinear coordinates is not what you may think in a simple minded way!**[2] When you see the gradient of a vector or the Laplacian of a vector, use vector identities to get rid of them until only the gradient of a scalar, divergence of a

[2] this is because $\mathbf{e}_i$ are *not* constant vectors, but can depend on ξ_i.

vector and curl of a vector are involved. Then use the above formulas for these three entities.

In deriving partial differential equations of fluid mechanics, the concept of conservation is closely associated with the divergence operator. The *substantial derivative* D/Dt needs the help of eq.(14) and the viscous stress term needs the help of eq.(15) to deal with curvilinear orthogonal coordinates. You need to know the derivation of, and be comfortable with, the *Reynold's Transport Theorem*:

$$\frac{d}{dt}\int\int\int_{\text{Volume}(t)} \phi \; d\upsilon \quad = \quad \int\int\int_{\text{Volume}} \frac{\partial \phi}{\partial t} \; d\upsilon \tag{35}$$

$$+ \int\int_{\text{Surface}} \phi \mathbf{n} \cdot \mathbf{V} \; d\sigma. \tag{36}$$

where Volume(t) on the left hand side emphasizes that the chosen volume of interest is being convected by the velocity field $\mathbf{V}$ and is therefore a function of time. (Loosely speaking, the d/dt on the left hand side can be interpreted as a Lagrangian derivative.)

Using the divergence theorems, conservation laws can be derived elegantly, without working tediously on each of the six surfaces of little cubes. You just need to know what the conservation law in English, and express the thought using volume and surface integrals. When you are done your derived integral form of the equation should say exactly what the original English sentence was meant to say.

When manipulating equations, always:

- check dimensional consistency,
- tensor index consistency,

You surely know what is a "material volume" and what is a "control volume." Using common sense to check obvious sign errors.

7 Substantial Derivative

Given a scalar field $\phi(\mathbf{x}, t)$. Let $\mathbf{x}(t)$ denote the trajectories of any chosen material "parcel" (tiny bit) of fluid. So, the velocity of the fluid parcel at $\mathbf{x}(t)$ is $d\mathbf{x}(t)/dt$. So what is the time rate of change of ϕ following a chosen material parcel of fluid? We take the time derivative of $\phi(\mathbf{x}, t)$ using the chain-rule (of calculus):

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \phi}{\partial t} + \mathbf{V} \cdot \frac{\partial \phi}{\partial \mathbf{x}} \tag{37}$$

This time derivative (taken while holding the identify of the fluid parcel fixed) is commonly known as the *substantial derivative*, and is usually denoted by D/Dt:

$$\frac{D\phi}{Dt} \equiv \frac{\partial\phi}{\partial t} + \mathbf{V} \cdot \frac{\partial\phi}{\partial\mathbf{x}}. \tag{38}$$

So, whenever you see something that looks like the right hand side of eq.(38), you know what that mathematical expression is saying to you in English.

8 Homework

1 : You can now prove the identities eq.(12)—eq.(15), and eq.(22). (Hint: always start with the double $\times$ term.)

2 : What is (h_1, h_2, h_3) in cylindrical coordinates (r, θ, z)?

3 : What is $\nabla^2\phi$ in cylindrical coordinates?

4 : What is (h_1, h_2, h_3) in spherical coordinates (R, θ, ψ)?

5 : What is $\nabla^2\phi$ in spherical coordinates?

6 : What are the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ components $\mathbf{A} \cdot \nabla\mathbf{A}$ in cylindrical polar coordinates? You can do it using eq.(14), or by brute force, noting that $\partial\mathbf{e}_r/\partial\theta = \mathbf{e}_\theta$ and $\partial\mathbf{e}_\theta/\partial\theta = -\mathbf{e}_r$ (OK? Can you derive this?).

7 : What is the $\mathbf{e}_\theta$ component of $\nabla^2\mathbf{A}$ in cylindrical polar coordinates? (It is *NOT* $\nabla^2 A_r$) Do it using eq.(15).

8 : You are given a volume in a fluid flow field. Write down the mass, momentum, thermal energy, kinetic energy, pollutant mass, and gold fish in this volume, using the notations in these notes.

9 : You may recall in your previous fluids course that the continuity equation is:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{V} = 0. \tag{39}$$

You have a soup with meat balls in it. What would you use for the continuity equation for this soup?