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Singular Perturbation Problems with Nearly Resonant Operators

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Dedicated to Nicholas, with fond memories of my year at Cornell.

I. Introduction

Interesting solution techniques for generating approximate solutions to problems containing a small parameter ε in some non-trivial way are called singular perturbation techniques. These techniques are particularly useful for non-linear problems for which no other analytical methods are available. For certain classes of problems, the appropriate techniques have been formalized into useful standard procedures [1, 2]. In many cases these procedures are not protected by general theorems; their validity is usually inferred from the reasonableness and self-consistency of the results and from experiences accumulated from numerous successful test cases.

In the present paper, we shall consider an interesting example which on first sight appears to be a routine problem solvable by the method of Matched-Asymptotic Expansions [1] (MAE). However, on closer examination various difficulties emerge. We shall show that the observed difficulties for this specific example is characteristic of a class of singular perturbation problems.

Consider the following convection-diffusion problem for $\phi(x, t; \varepsilon)$:

$$\frac{\partial \phi}{\partial t} = \lambda(\varepsilon) P(\phi, x) - \mathbf{L}(\varepsilon)\phi \tag{1.1}$$

where the operator $\mathbf{L}(\varepsilon)$ is defined by

$$\mathbf{L}(\varepsilon)\phi = \frac{\partial \phi}{\partial x} - \varepsilon \frac{\partial^2 \phi}{\partial x^2}, \tag{1.2a}$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{x=0} = 0, \quad (\phi)_{x=1} = 0, \tag{1.2b}$$

and $P(\phi, x)$ is some arbitrary function of order unity. The parameter ε is small and positive. We shall consider the particular choice of $\lambda(\varepsilon)$ given by

$$\lambda(\varepsilon) = \frac{1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon}\right) \tag{1.3}$$

so that it is extremely small. For example, when $\varepsilon = 10^{-2}$, $\lambda(10^{-2})$ is less than 10^{-41}. The initial condition is given by

$$\phi(x, 0; \varepsilon) = I(x), \quad 0 \le x \le 1, \tag{1.4}$$

where $I(x)$ is of order unity. The objective here is to find the large t and steady state solutions for arbitrarily given $P(\phi, x)$ and $I(x)$.

The problem as stated appears deceptively simple. Except for the potential non-linearities in $P(\phi, x)$ and the rather curious choice of $\lambda(\varepsilon)$, the problem seems quite non-controversial and ideally suited for MAE. Let us first consider the steady state solution, $\phi_\infty(x; \varepsilon) = \phi(x, \infty; \varepsilon)$. Using standard MAE procedures, one would obtain

$$\phi_\infty(x; \varepsilon) \simeq A\left[1 - \exp\left(\frac{1-x}{\varepsilon}\right)\right] \tag{1.5}$$

where the amplitude A appears as an arbitrary constant. If one now solves the initial-value problem by standard MAE procedures, one would obtain $A = I(0)$ for $t > 1$. In both analyses the exponentially small term did not participate. At this point one might be led to conclude (erroneously) that the steady state solution is controlled by $I(0)$ and is independent of $P(\phi, x)$. Examining further, one finds that if $\lambda(\varepsilon)P(\phi, x)$ is formally neglected, the resulting linear steady state equation can be solved exactly, yielding $A = 0$. If one considers another test case with $P(\phi, x) = a\phi$, the resulting linear steady state equation can again be solved exactly, showing that A is arbitrary only if $a = 1$. Otherwise, no non-trivial steady state solution exists. Thus, contradictory results have been obtained by several seemingly justifiable procedures, a situation particularly vexing because MAE has in the past been most successful with convection-diffusion problems. The two linear test cases ($P = 0$, $P = a\phi$) clearly indicate that, in spite of the smallness of $\lambda(\varepsilon)$, the steady state solution is dependent on $P(\phi, x)$. For problems with arbitrarily given non-linear $P(\phi, x)$ and $I(x)$, the situation at this point is far from clear.

II. Preliminary Discussion

For the given example, various modifications of standard MAE procedures may be suggested to overcome the observed difficulties. We shall show that these observed difficulties are characteristic of a class of singular perturbation problems and can be identified by studying the properties of the operator $\mathbf{L}(\varepsilon)$. A general procedure is then suggested to deal with this class of problems.

Because of the powerfulness of standard singular perturbation procedures for non-linear problems, the linear concepts of eigenfunction expansions are often overlooked when a small parameter ε exists. From the point of view of eigenfunctions, the observed apparent difficulties are immediately clear [3]. The operator $\mathbf{L}(\varepsilon)$ defined by Eqs. (1.2a) *and* (1.2b) is a *nearly-resonant* operator in the sense that its lowest eigenvalue is asymptotically small in the limit as $\varepsilon \to 0$. Hence $\phi(x, t; \varepsilon)$ at large t and small ε evolves asymptotically slowly with t and is extremely sensitive to small forcing terms.

Once the role of a nearly-resonant $\mathbf{L}(\varepsilon)$ has been identified, numerous examples can be found where standard singular perturbation procedures would encounter similar difficulties. The particular example was chosen to illustrate the point because of the rather dramatic smallness of $\lambda(\varepsilon)$ and because the solutions obtained by standard MAE procedures appear indeed on cursory examinations to be reasonable and self-consistent.

III. Eigenfunction Expansion

Let us consider the class of initial-value problem posed by Eqs. (1.1) and (1.4) where $\mathbf{L}(\varepsilon)$ is a linear nearly-resonant operator with properties to be defined.

Let $\bar{\mathbf{L}}(\varepsilon)$ denote the adjoint of $\mathbf{L}(\varepsilon)$. We shall define $Y_i(x;\varepsilon)$, $y_i(x;\varepsilon)$, and $\Lambda_i(\varepsilon)$ by

$$\mathbf{L}(\varepsilon)Y_i = \Lambda_i Y_i, \tag{3.1a}$$

$$\bar{\mathbf{L}}(\varepsilon)y_i = \Lambda_i y_i, \tag{3.1b}$$

where the eigenvalues $\Lambda_i(\varepsilon)$'s are assumed real and are ordered in ascending magnitudes. We shall normalize the eigenfunctions $Y_i(x;\varepsilon)$'s by the requirement that their maximum magnitude be of order unity:

$$|Y_i(x;\varepsilon)|_{\max} = 0(1), \quad i = 0, 1, 2, \dots. \tag{3.2}$$

The adjoint eigenfunction $y_i(x;\varepsilon)$'s are normalized by the ortho-normal relations:

$$[Y_i(x;\varepsilon)\cdot y_i(x;\varepsilon)] = \delta_{ij} \tag{3.3}$$

where δ_{ij} is the Kronecker delta and conventional notation for scalar product has been used.

The operator $\mathbf{L}(\varepsilon)$ is said to be a linear nearly-resonant operator if

(a) Y_i, y_i, and Λ_i exist for small but non-zero values of ε.
(b) All eigenvalues except $\Lambda_0(\varepsilon)$ are positive.
(c) $\Lambda_0(\varepsilon)$ is non-degenerate.
(d) The eigenfunctions are complete.
(e) A positive function $\Omega(\varepsilon)$ can be found such that

$$\left[1\cdot|y_i|\right] \leq \Omega(\varepsilon), \quad i \geq 1, \quad \varepsilon \neq 0. \tag{3.4}$$

(f) $\Lambda_0(\varepsilon)$ satisfied

$$\lim_{\varepsilon\to 0} \frac{\Lambda_0(\varepsilon)\Omega(\varepsilon)}{\Lambda_1(\varepsilon)} \to 0. \tag{3.5}$$

Conditions (a), (b), (c), and (d) are completely conventional. Condition (e) does not require that $\Omega(0)$ exists; in fact $\Omega(\varepsilon)$ may be singular as $\varepsilon \to 0$. Condition (f) clearly identifies the lowest mode to be an 'asymptotically resonant' mode.

The factor $\lambda(\varepsilon)$ in Eq. (1.1) shall be called the pumping factor. We shall consider only *non-interacting pumping factors* defined by

$$\lim_{\varepsilon \to 0} \frac{\lambda(\varepsilon)\Lambda_0(\varepsilon)}{\Lambda_1} \to 0.$$

Since the eigenfunctions are complete, we can formally expand $\phi(x, t; \varepsilon)$ as follows

$$\phi(x, t; \varepsilon) = \sum_{j=0}^{\infty} A_j(t; \varepsilon) Y_j(x; \varepsilon). \tag{3.7}$$

It is straightforward to show that after a transient period $t^*(\varepsilon)$, all A_j's except $A_0(t; \varepsilon)$ die out exponentially so that for $t \gg t^*(\varepsilon)$ the solution is dominated by the lowest mode:

$$\phi(x, t \gg t^*(\varepsilon), \varepsilon) \simeq A_0(t; \varepsilon) Y_0(x; \varepsilon). \tag{3.8}$$

The main tasks at hand are the estimation of $t^*(\varepsilon)$ and the determination of $A_0(t; \varepsilon)$.

Substituting Eq. (3.7) into Eq. (1.1) and taking the scalar product with $y_i(x; \varepsilon)$, we obtain

$$\frac{dA_i}{dt} = \lambda(\varepsilon)[P \cdot y_i] - \Lambda_i A_i, \quad i = 0, 1, 2, \ldots. \tag{3.9a}$$

The initial conditions are

$$A_i(0; \varepsilon) = [I \cdot y_i], \quad i = 0, 1, 2, \ldots. \tag{3.9b}$$

For non-interacting pumping factors, $A_i(t; \varepsilon)$ for $i \neq 0$ is given approximately by $A_i(0; \varepsilon) \exp(-\Lambda_i t)$. Since $y_i(x; \varepsilon)$ may be singular in the limit as $\varepsilon \to 0$, the upper bound of $|A_i(0; \varepsilon)|$ is estimated by $\Omega(\varepsilon)|I|_{\max}$. Thus we have

$$|A_i(t; \varepsilon)| \leq |I|_{\max} \exp[-\Lambda_i(\varepsilon)(t - t^*(\varepsilon))], \quad i \neq 0 \tag{3.10}$$

where

$$t^*(\varepsilon) = \frac{\ln \Omega(\varepsilon)}{\Lambda_1(\varepsilon)}. \tag{3.11}$$

Hence, for $t \gg t^*$, Eq. (3.8) holds and $P(\phi, x)$ is given approximately by $P(A_0 Y_0, x)$. Note that if $\lambda(\varepsilon)$ is not a non-interacting pumping factor, the various modes will interact (when $P(\phi, x)$ is non-linear) and all the A_i's will be coupled.

The governing equation for $A_0(t; \varepsilon)$ is then

$$\frac{dA_0}{dt} = \lambda(\varepsilon)[P(A_0 Y_0, x) \cdot y_0] - \Lambda_0(\varepsilon) A_0. \tag{3.12a}$$

The initial condition for $A_0(t;\varepsilon)$ is

$$A_0(0;\varepsilon) = [I \cdot y_0]. \tag{3.12b}$$

Note that once $\mathbf{L}(\varepsilon)$ has been shown to be a linear nearly-resonant operator, the large t solutions for Eq. (1.1) with non-interacting pumping factor and arbitrarily given $P(\phi, x)$ and $I(x)$ can be constructed from knowledge of $\Lambda_0(\varepsilon)$, $Y_0(x;\varepsilon)$, and $y_0(x;\varepsilon)$ alone. Standard singular perturbation techniques may be used to find $Y_0(x;\varepsilon)$ and $y_0(x;\varepsilon)$. The determination of $\Lambda_0(\varepsilon)$, however, requires some care as will be seen later.

IV. Application to the Given Example

The adjoint operator $\bar{\mathbf{L}}(\varepsilon)$ to $\mathbf{L}(\varepsilon)$ defined by Eqs. (1.2) is

$$\bar{\mathbf{L}}(\varepsilon) y_i = -\frac{\partial y_i}{dx} - \varepsilon \frac{\partial^2 y_i}{\partial x^2}, \tag{4.1a}$$

$$\left(\varepsilon \frac{\partial y_i}{\partial x} + y_i\right)_{x=0} = 0 \left(y_i\right)_{x=1} = 0. \tag{4.1b}$$

The lowest eigenfunction $Y_0(x;\varepsilon)$ and adjoint eigenfunction $y_0(x;\varepsilon)$ can be obtained by solving Eqs. (3.1) using standard MAE procedures. We obtain

$$Y_0(x;\varepsilon) \simeq 1 - \exp\left(\frac{1-\mathrm{x}}{\varepsilon}\right), \quad \varepsilon \ll 1, \tag{4.2a}$$

$$y_0(x;\varepsilon) = \frac{1}{\varepsilon} \exp\left(-\frac{x}{\varepsilon}\right) Y_0(x;\varepsilon). \tag{4.2b}$$

The eigenvalue $\Lambda_0(\varepsilon)$ in general satisfies

$$\Lambda_0(\varepsilon) = \frac{[y_0 \cdot \mathbf{L} Y_0]}{[y_0 \cdot Y_0]}. \tag{4.3}$$

Using Eq. (4.sb) for y_0, we have

$$\Lambda_0(\varepsilon) = \frac{\varepsilon \int_0^1 e^{-x/\varepsilon} \left(\frac{dY_0}{dx}\right)^2 dx}{\int_0^1 e^{-x/\varepsilon} Y_0^2 \, dx}. \tag{4.4a}$$

Using Eq. (4.2a) for $Y_0(x;\varepsilon)$, we obtain

$$\Lambda_0(\varepsilon) \simeq \frac{1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon}\right), \quad \varepsilon \ll 1. \tag{4.4b}$$

The rest of the eigenfunctions and eigenvalues are easily found:

$$Y_i(x;\varepsilon) = \exp\left(\frac{x-1}{2\varepsilon}\right)\sin\,[k_i(x-1)],$$

$$y_i(x;\varepsilon) = \frac{2}{1+2\varepsilon\cos^2 k_i}\exp\left(\frac{1-x}{\varepsilon}\right)Y_i(x;\varepsilon), \tag{4.5}$$

$$\Lambda_i(\varepsilon) = \frac{1}{4\varepsilon\cos^2 k_i} < \frac{1}{4\varepsilon}, \quad i = 1, 2, 3, \ldots.$$

where

$$\tan k_i = 2\varepsilon k_i > 0, \quad i = 1, 2, 3, \ldots. \tag{4.6}$$

Note that all $y_i(x;\varepsilon)$'s are singular in the limit as $\varepsilon \to 0$. By inspection, we can choose $\Omega(\varepsilon)$ to be

$$\Omega(\varepsilon) = 4\varepsilon \exp\frac{1}{2\varepsilon}\cdot \tag{4.7}$$

We can readily verify that $\mathbf{L}(\varepsilon)$ defined by Eqs. (1.2) is indeed a linear nearly-resonant operator and $\lambda(\varepsilon)$ given by Eq. (1.3) is a non-interacting pumping factor. From Eq. (3.11), we obtain

$$t^*(\varepsilon) = 2 + 4\varepsilon \ln 4\varepsilon. \tag{4.8}$$

Thus the higher modes are damped out when $t > 2$. To concretely illustrate the determination of $A_0(t;\varepsilon)$, consider the specific function $P(\phi, x)$ below:

$$P(\phi, x) = -\phi + 3\phi^2 - (\phi\cos x)^3. \tag{4.9}$$

Eqs. (3.12) become

$$\frac{dA_0}{dt} \simeq \Lambda_0(\varepsilon)A_0(A_0-1)(A_0-2), \tag{4.10a}$$

$$A_0(0;\varepsilon) \simeq I(0), \quad \varepsilon \ll 1 \tag{4.10b}$$

where we have taken advantage of the fact that for $\varepsilon \ll 1$, $y_0(x;\varepsilon)$ is essentially a delta function at $x = 0$. It is seen that $A_0(t;\varepsilon)$ evolves extremely slowly with t, approaching 0 or 2 as $t \to \infty$ depending on whether $I(0)$ is less than or greater than 1.0. Any other $P(\phi, x)$, $I(x)$, and non-interacting $\lambda(\varepsilon)$ can be studied similarly without difficulties.

V. Discussion

In terms of eigenfunction concepts, the procedure employed here is completely straightforward except for two minor complications. Firstly, the adjoint eigenfunctions $y_i(x;\varepsilon)$'s may be singular in the limit as $\varepsilon \to 0$. The estimate of the time required for the higher modes to damp out is seen to be dependent on $\Omega(\varepsilon)$. Secondly, the evaluation of $\Lambda_0(\varepsilon)$ requires some care; otherwise, one would merely obtain the correct but non-useful conclusion that $\Lambda_0(\varepsilon)$ is nearly zero. In the example, $\Lambda_0(\varepsilon)$ is first expressed exactly in Eq. (4.4a) in positive-definite form and is then evaluated using approximate $Y_0(x;\varepsilon)$ obtained by standard MAE procedures. Since $\Lambda_0(\varepsilon)$ is asymptotically small, in a general problem its accurate determination remains the main task to be resolved.

Generally, the appropriate standard singular perturbation procedure can be used to construct $Y_0(x;\varepsilon)$ and $y_0(x;\varepsilon)$. In fact, it is clear that $Y_0(x;\varepsilon)$ and $y_0(x;\varepsilon)$ can be obtained by solving $\mathbf{L}(\varepsilon)Y_0 \simeq 0$ and $\bar{\mathbf{L}}(\varepsilon)y_0 = 0$ *approximately* in spite of the fact that exact solutions to $\mathbf{L}(\varepsilon)Y_0 = 0$ and $\bar{\mathbf{L}}(\varepsilon)y_0 = 0$ may not exist. The various apparent difficulties in our example are thus typical 'symptoms' of singular perturbation problems with nearly-resonant operators. In the language of numerical analysis, this class of problems are asymptotically 'stiff'. If the example were solved on a computer, one would readily 'verify' the results obtained by MAE for $t = 0(1)$. It is probably impossible to obtain numerical solutions for $\Lambda_0(\varepsilon)t = 0(1)$ when ε is moderately small.

The basic ideas employed here can be generalized when $\mathbf{L}$ is a non-linear operator, $\mathbf{L} = \mathbf{U}(\phi;\varepsilon)$. We may write formally

$$\phi(x,t;\varepsilon) = Z(x;A,\varepsilon) + \psi(x,t;\varepsilon) \tag{5.1}$$

where $Z(x;A,\varepsilon)$ is some unknown function depending on t only through $A(t;\varepsilon)$. Linearing the given problem about $Z(x;A,\varepsilon)$, we can obtain a linear problem for $\psi(x,t;\varepsilon)$. By requiring that $\psi(x,t;\varepsilon)$ be uniformly small for all t, an analogous analysis will yield not only the equations for $Z(x;A,\varepsilon)$ and $A(t;\varepsilon)$, but also the definition of a non-linear nearly resonant operator. The details of this generalization are beyond the scope of the present paper.

References

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Abstract

A simple convection-diffusion problem is used as an example for a class of singular perturbation problems for which application of standard singular perturbation procedures could lead to misleading results. The properties of the operator appearing in the problem are identified and a solution technique using eigenfunction concepts is presented.

Zusammenfassung

Ein einfaches Problem mit Konvektion und Diffusion wird als Beispiel für singuläre Störungsprobleme gegeben, für welche Standardmethoden zu falschen Resultaten führen können. Die Eigenschaften des Operators, der im Problem auftritt, werden untersucht und eine Lösungsmethode wird gegeben, die auf Eigenfunktionen beruht.

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