Lecture 01: One Period Model

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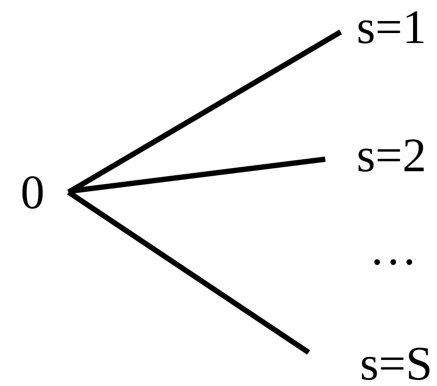
Overview

1. Securities Structure
   - Arrow-Debreu securities structure
   - Redundant securities
   - Market completeness
   - Completing markets with options
2. Pricing (no arbitrage, state prices, SDF, EMM …)
3. Optimization and Representative Agent (Pareto efficiency, Welfare Theorems, …)

The Economy

- State space (Evolution of states)
  - Two dates: $t=0,1$
  - S states of the world at time $t=1$



- Preferences
  - $U(c_0, c_1, \ldots, c_S)$
  - $MRS^A_{s,0} = -\frac{\partial U^A / \partial c^A_s}{\partial U^A / \partial c^A_0}$ (slope of indifference curve)
- Security structure
  - Arrow-Debreu economy
  - General security structure

Security Structure

- Security *j* is represented by a payoff *vector*

$$\left(x_1^j, x_2^j, \ldots, x_S^j\right)$$

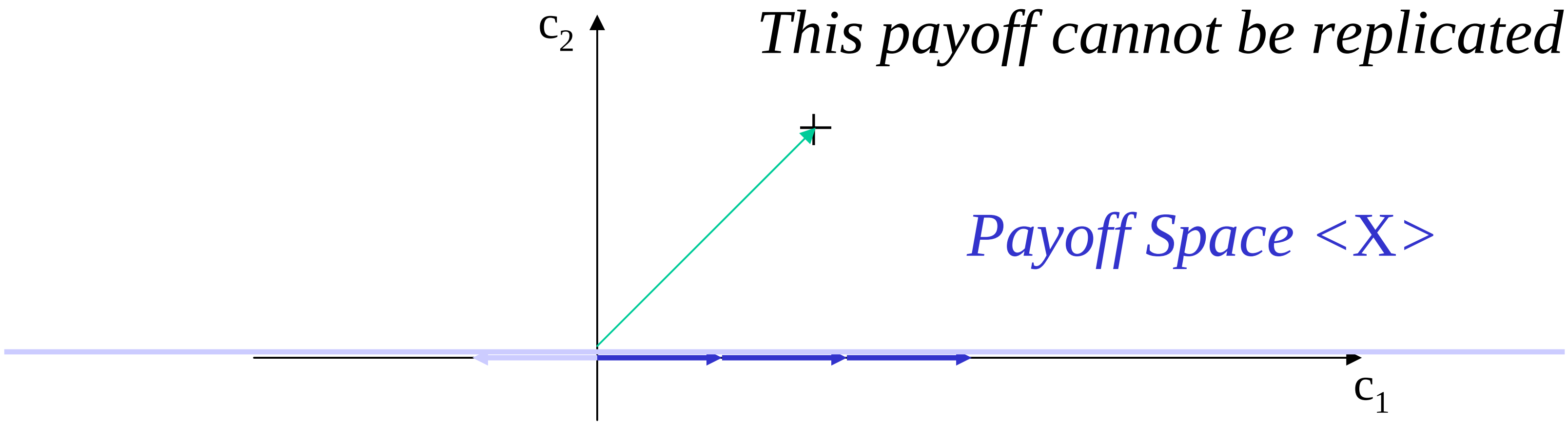
- Security structure is represented by payoff matrix

$$X = \begin{pmatrix} x_1^j & x_2^j & \cdots & x_{S-1}^j & x_S^j \\ x_1^{j+1} & x_2^{j+1} & \cdots & x_{S-1}^{j+1} & x_S^{j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{J-1} & x_2^{J-1} & \cdots & x_{S-1}^{J-1} & x_S^{J-1} \\ x_1^J & x_2^J & \cdots & x_{S-1}^J & x_S^J \end{pmatrix}$$

- NB. Most other books use the transpose of X as payoff matrix.

Arrow-Debreu Security Structure in R^2

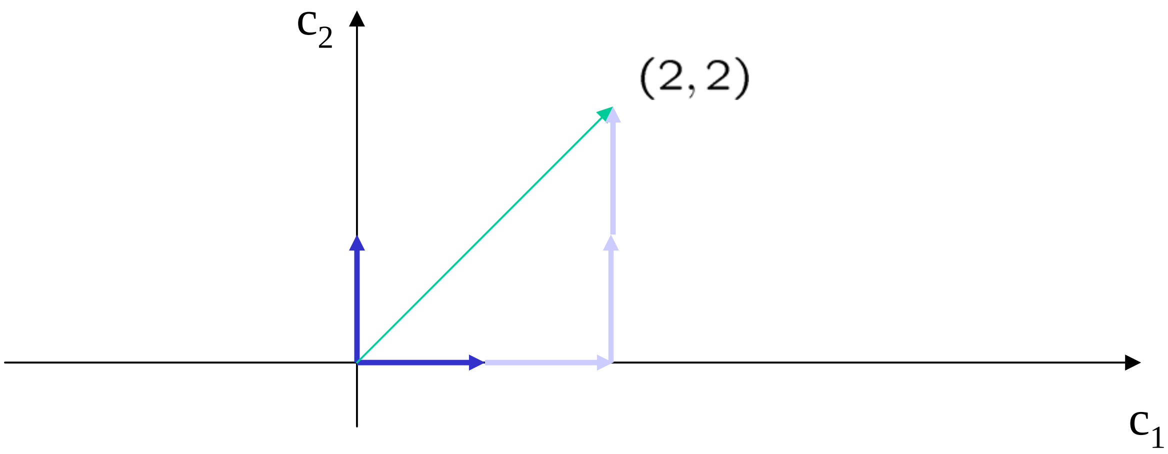
One A-D asset $e_1 = (1,0)$



⇒ Markets are **incomplete**

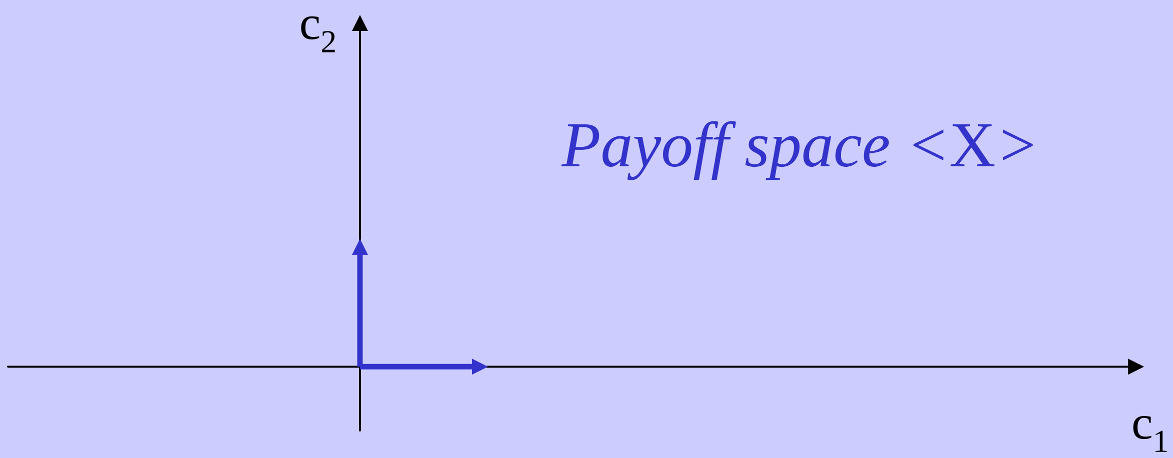
Arrow-Debreu Security Structure in R^2

Add second A-D asset $e_2 = (0,1)$ to $e_1 = (1,0)$



Arrow-Debreu Security Structure in R^2

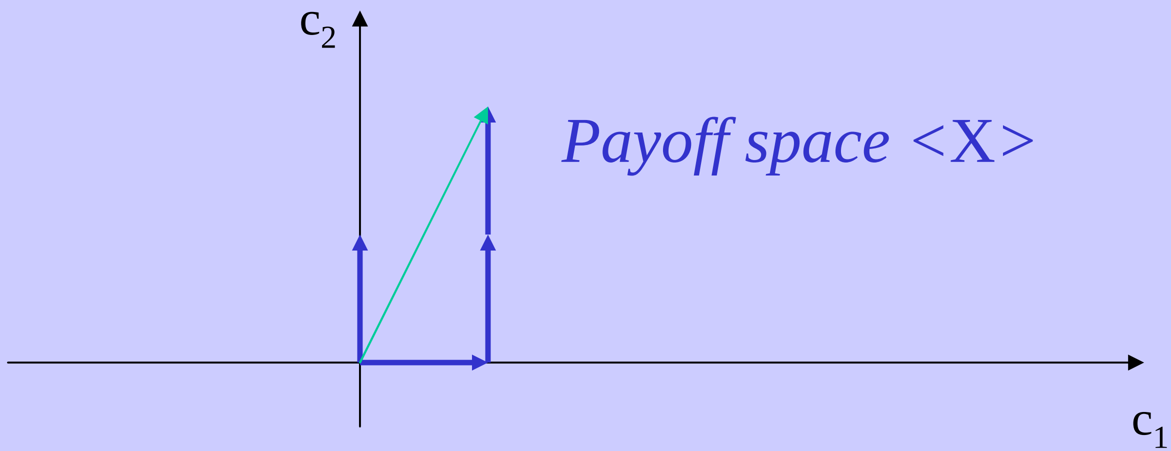
Add second A-D asset $e_2 = (0,1)$ to $e_1 = (1,0)$



Any payoff can be replicated with two A-D securities

Arrow-Debreu Security Structure in R^2

Add second asset (*1,2*) to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



New asset is ***redundant*** *– it does not enlarge the payoff space*

Arrow-Debreu Security Structure

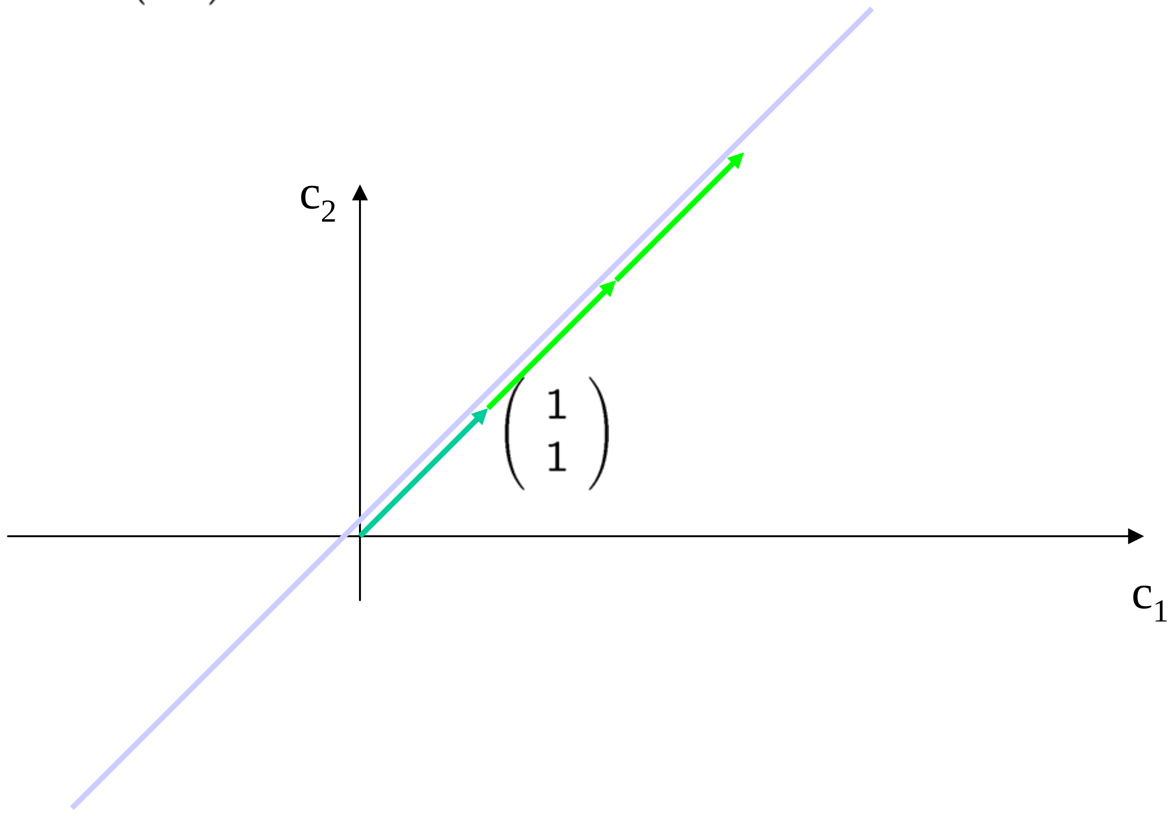
$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

- S Arrow-Debreu securities
- each state s can be insured individually
- All payoffs are linearly independent
- Rank of $X = S$
- Markets are complete

General Security Structure

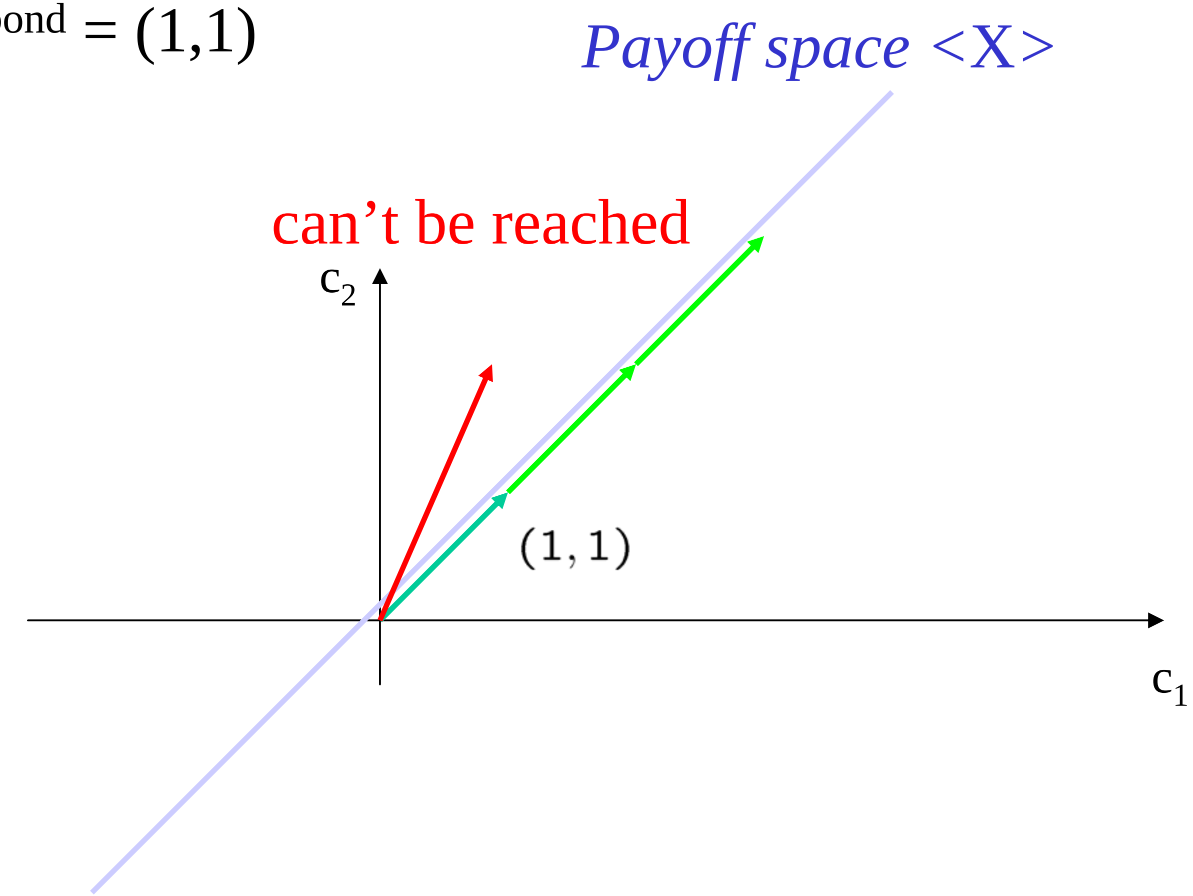
Only bond $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Payoff space <X>



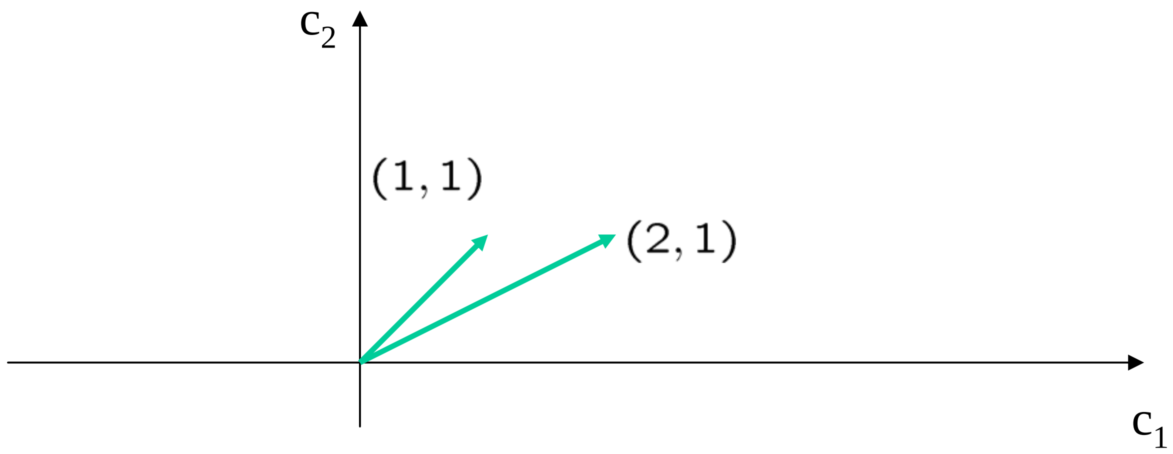
General Security Structure

Only bond $x^{bond} = (1,1)$



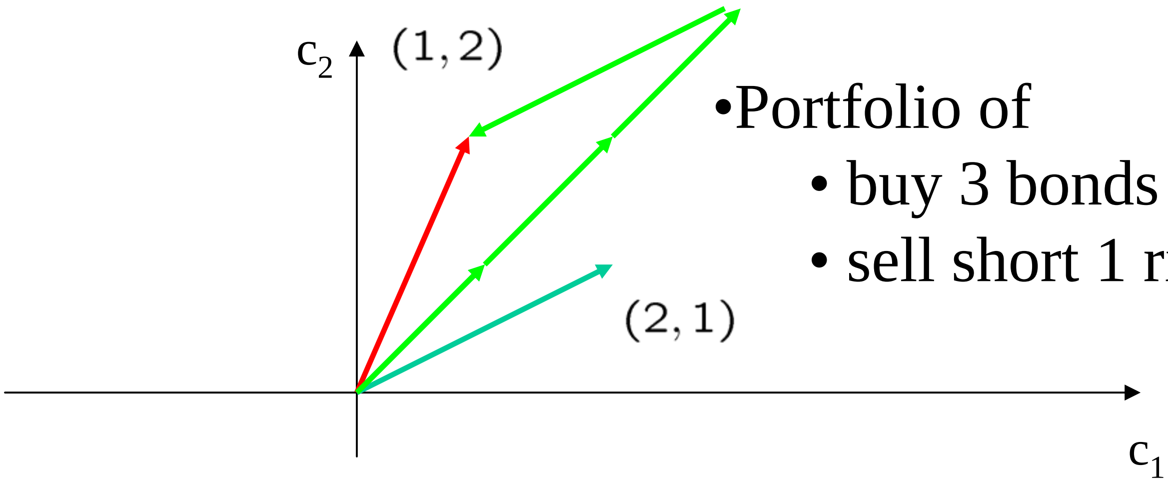
General Security Structure

Add security (*2*,*1*) to bond (*1*,*1*)



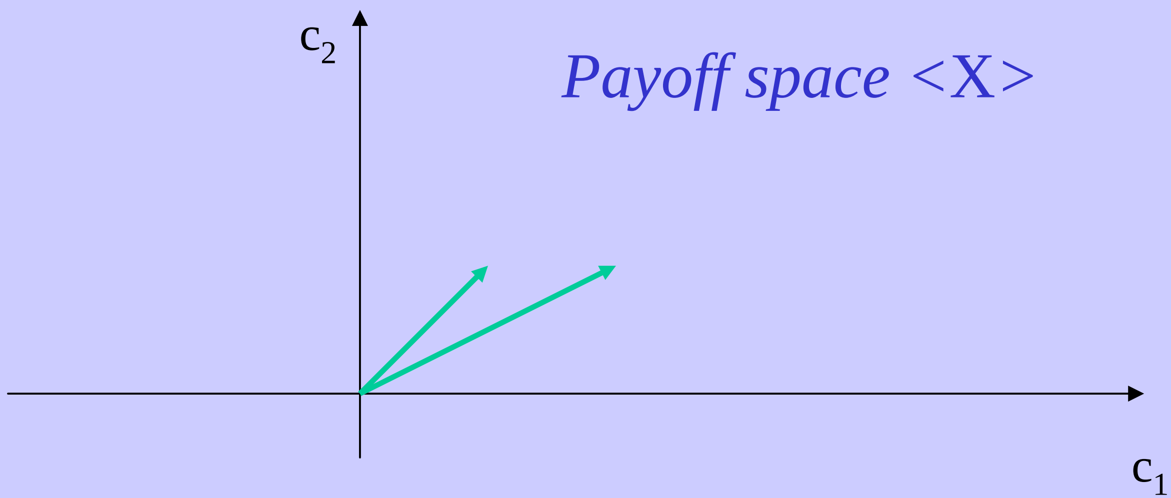
General Security Structure

Add security (*2*,*1*) to bond (*1*,*1*)



- Portfolio of
  - buy 3 bonds
  - sell short 1 risky asset

General Security Structure



Two assets ***span*** *the payoff space*

Market are complete with security structure $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Payoff space coincides with payoff space of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

General Security Structure

- Portfolio: vector $h \in R^J$ (quantity for each asset)
- Payoff of Portfolio h is $\sum_j h^j x^j = h'X$
- Asset span

$$< X > = \{z \in \mathbb{R}^S : z = h'X \text{ for some } h \in \mathbb{R}^J\}$$

  - <X> is a linear subspace of R^S
  - Complete markets <X> = R^S
  - Complete markets if and only if $rank(X) = S$
  - Incomplete markets $rank(X) < S$
  - Security *j* is redundant if $x^j = h'X$ with $h^j=0$

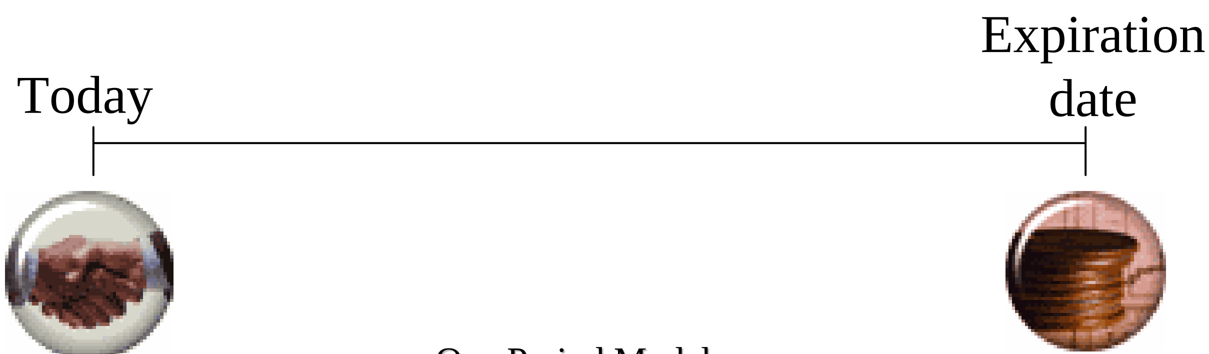
Introducing derivatives

- Securities: property rights/contracts
- Payoffs of derivatives *derive* from payoff of underlying securities
- Examples: forwards, futures, call/put options

- Question:
  Are derivatives necessarily redundant assets?

Forward contracts

- Definition: A binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today
- Futures contracts are same as forwards in principle except for some institutional and pricing differences
- A forward contract specifies:
  - ❑ The features and quantity of the asset to be delivered
  - ❑ The delivery logistics, such as time, date, and place
  - ❑ The price the buyer will pay at the time of delivery



Reading price quotes

Index futures

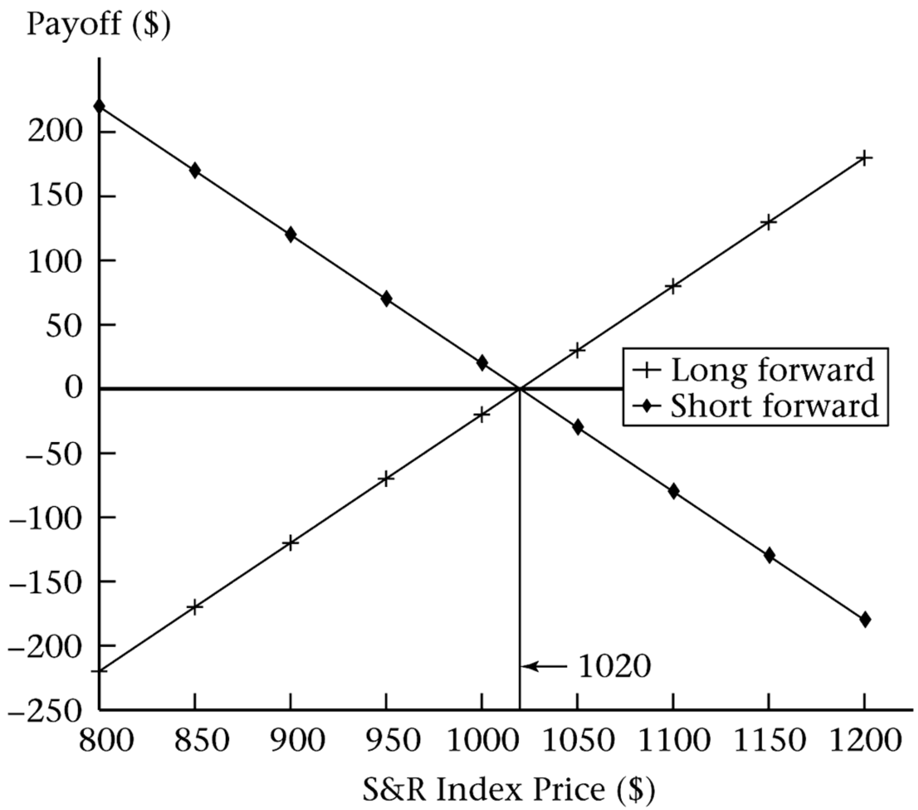
- Settlement price (last transaction of the day)
- *Low* of the day
- *High* of the day
- The *open* price
- Expiration month
- Daily change
- Lifetime *high*
- Lifetime *low*
- Open interest

INDEX

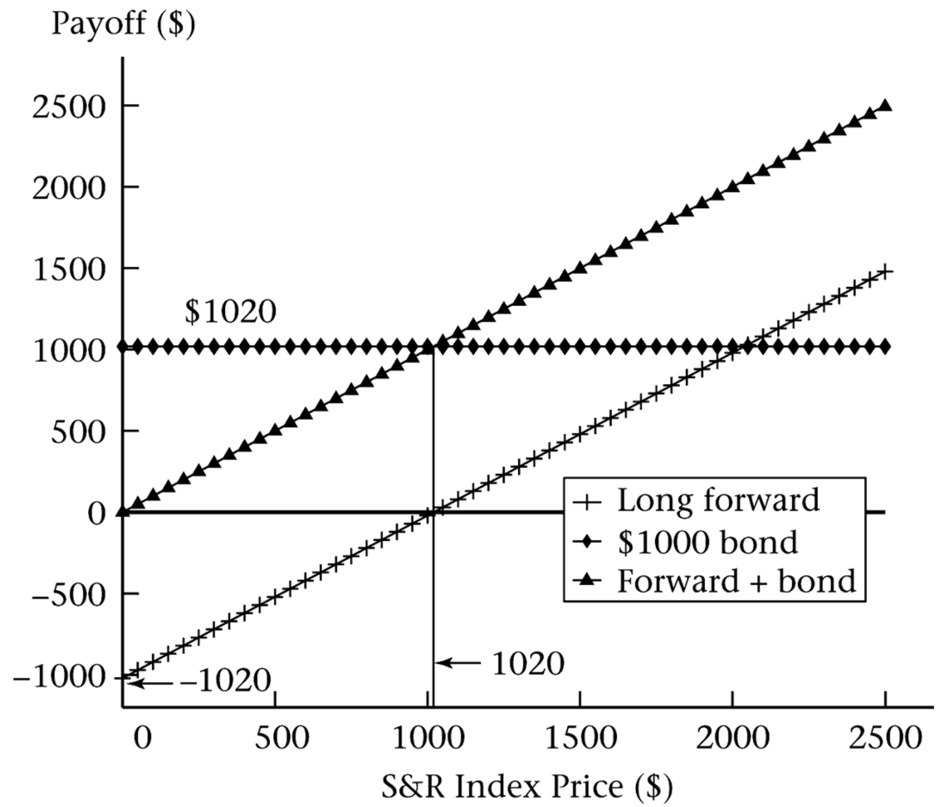
	OPEN	HIGH	LOW	SETTLE	CHANGE	LIFETIME HIGH	LIFETIME LOW	OPEN INT.
DJ Industrial Average (CBOT)-$10 times average								
Mar	9891	9902	9655	9683	– 224	11150	7900	27,474
June	9865	9890	9665	9688	– 228	10951	9080	589
Est vol 21,000; vol Fri 17,070; open int 28,254, +822.								
Idx prl: Hi 9905.46; Lo 9677.54; Close 9687.09, −220.17.								
S&P 500 Index (CME)-$250 times index								
Mar	112350	112370	109100	109530	– 2810	134960	94100	474,811
June	111950	111950	109350	109730	– 2830	170550	95030	17,224
Dec	111580	111580	110020	110390	– 2930	150070	96130	304
Est vol 79,914; vol Fri 65,250; open int 502,626, −701.								
Idx prl: Hi 1122.20; Lo 1092.25; Close 1094.44, −27.76.								
Mini S&P 500 (CME)-$50 times index								
Mar	112325	112400	109100	109525	– 2825	117850	99850	100,297
Vol Fri 193,620; open int 100,323, −4,791.								
S&P Midcap 400 (CME)-$500 times index								
Mar	502.70	504.00	492.75	493.95	– 10.95	560.00	412.95	13,453
Est vol 1,140; vol Fri 1,101; open int 13,453, −207.								
Idx prl: Hi 504.26; Lo 492.74; Close 493.38, −10.88.								
Nikkei 225 Stock Average (CME)-$5 times index								
Mar	9690.	9700.	9555.	9580.	– 130	14620.	9245.	15,750
Est vol 667; vol Fri 2,100; open int 15,817, −17.								
Idx prl: Hi 9809.82; Lo 9623.99; Close 9631.93, −159.50.								
Nasdaq 100 (CME)-$100 times index								
Mar	152900	153550	147300	148700	– 4800	189400	112000	51,803
Est vol 18,215; vol Fri 17,500; open int 51,812, +763.								
Idx prl: Hi 1528.30; Lo 1471.52; Close 1479.17, −48.98.								

Payoff diagram for forwards

- Long and short forward positions on the S&R 500 index:



Forward vs. outright purchase



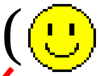
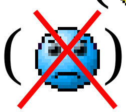
- Forward + bond = Spot price at expiration - \$1,020 (Forward payoff) + \$1,020 (Bond payoff)

  = Spot price at expiration

Additional considerations (ignored)

- Type of settlement
  - ❑ Cash settlement: less costly and more practical
  - ❑ Physical delivery: often avoided due to significant costs
- Credit risk of the counter party
  - ❑ Major issue for over-the-counter contracts
    - Credit check, collateral, bank letter of credit
  - ❑ Less severe for exchange-traded contracts
    - Exchange guarantees transactions, requires collateral

Call options

- A non-binding agreement (right but not an obligation) to buy an asset in the future, at a price set today
- Preserves the upside potential (), while at the same time eliminating the unpleasant () downside (for the buyer)
- The seller of a call option is obligated to deliver if asked

Definition and Terminology

- A call option gives the owner the right but not the obligation to buy the underlying asset at a predetermined price during a predetermined time period
- Strike (or exercise) price: The amount paid by the option buyer for the asset if he/she decides to exercise
- Exercise: The act of paying the strike price to buy the asset
- Expiration: The date by which the option must be exercised or become worthless
- Exercise style: Specifies when the option can be exercised


European-style: can be exercised only at expiration date


American-style: can be exercised at any time before expiration


Bermudan-style: can be exercised during specified periods

Reading price quotes

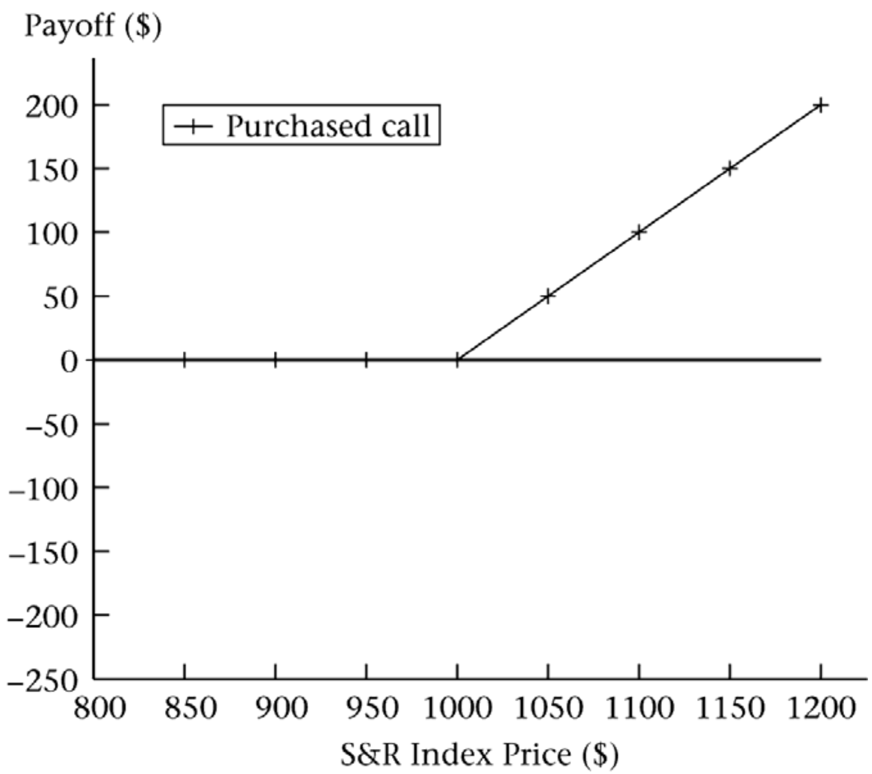
S&P500 Index options

Strike price

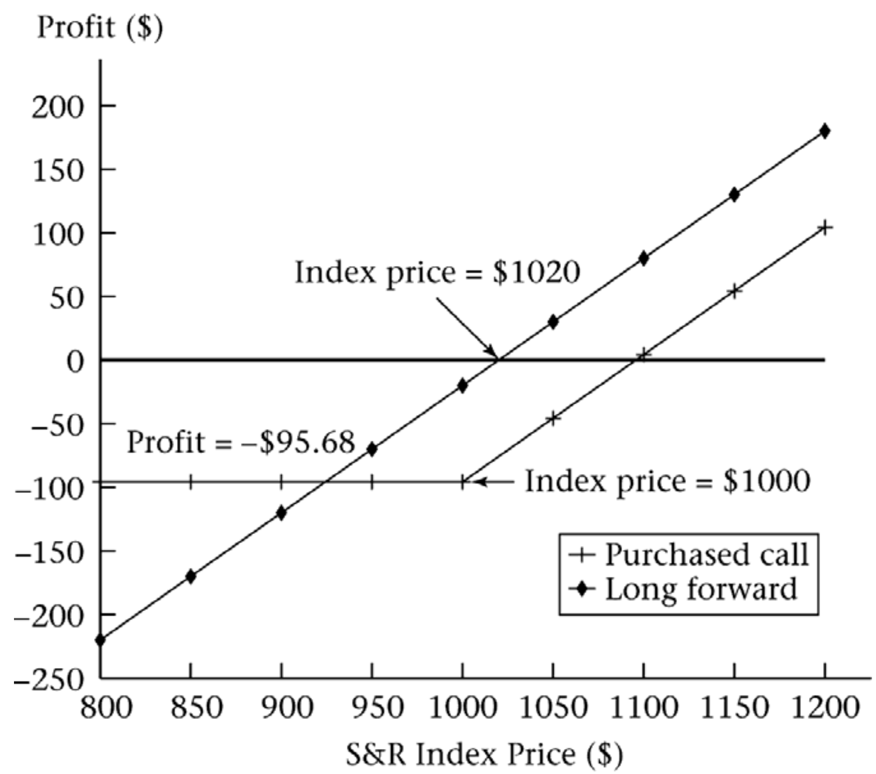
STRIKE		VOL.	LAST	NET CHG.	OPEN INT.
		S & P 500(SPX)			
Feb	1080 c	100	26.50	...	...
Feb	1080 p	358	13	+ 8.00	5
Mar	1080 c	10	44	...	...
Mar	1080 p	17	21.40	+ 6.00	412
Feb	1090 c	4	19	...	...
Feb	1090 p	141	15.80	+ 9.00	279
Mar	1090 c	270	32	...	302
Mar	1090 p	343	28	...	302
Feb	1100 c	1,041	15	– 16.20	6,763
Feb	1100 p	3,246	20.10	+ 11.80	26,497
Mar	1100 c	4,439	27	– 15.00	19,083
Mar	1100 p	8,235	33	+ 12.50	30,294
Apr	1100 c	81	37	– 15.00	1,728
Apr	1100 p	2,011	44	+ 14.00	4,126
Feb	1110 c	1,316	9	– 15.00	738
Feb	1110 p	1,032	27	+ 15.50	1,472
Feb	1120 c	805	6.30	– 9.80	1,057
Feb	1120 p	225	33.50	+ 18.50	1,626
Mar	1120 c	838	18	...	5,239
Mar	1120 p	953	43.50	...	5,095
Apr	1120 c	150	33.50	– 6.50	10

Diagrams for purchased call

- Payoff at expiration



- Profit at expiration



Put options

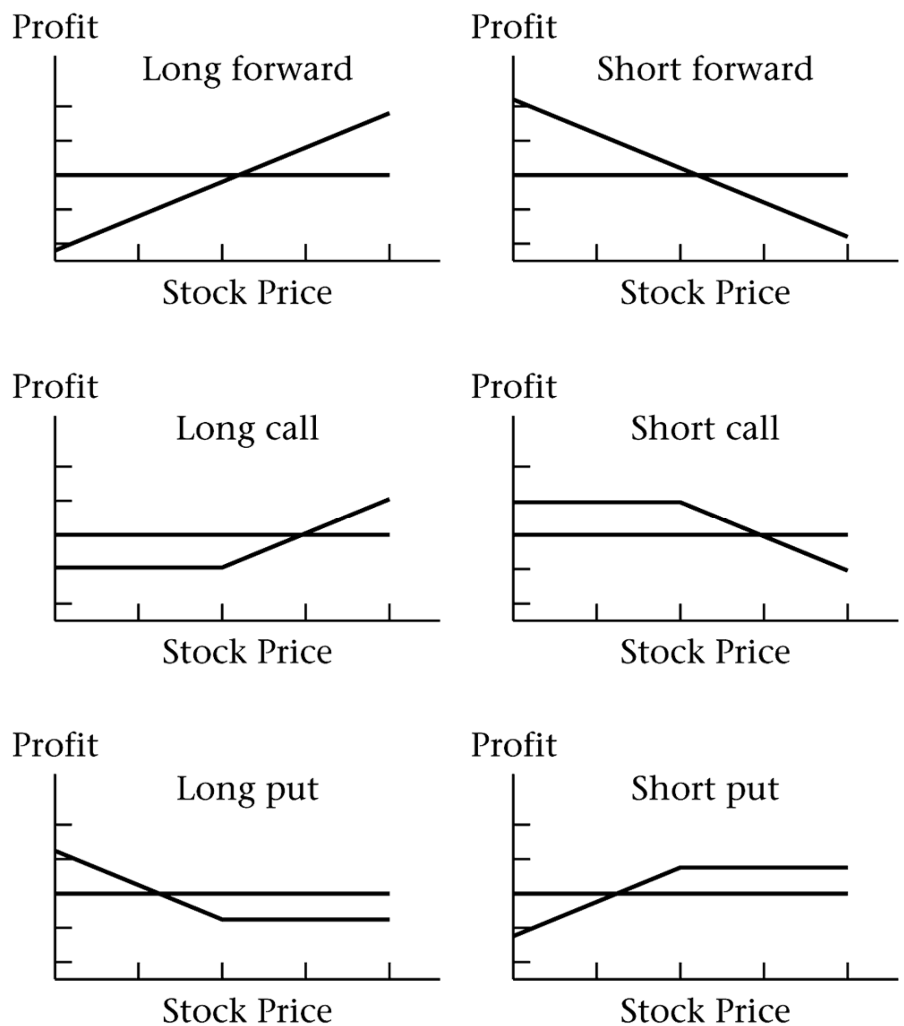
- A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period
- The seller of a put option is obligated to buy if asked
- Payoff/profit of a purchased (i.e., long) put:
  - Payoff = *max* [0, strike price – spot price at expiration]
  - Profit = Payoff – future value of option premium
- Payoff/profit of a written (i.e., short) put:
  - Payoff = - *max* [0, strike price – spot price at expiration]
  - Profit = Payoff + future value of option premium

A few items to note

- A call option becomes more profitable when the underlying asset appreciates in value
- A put option becomes more profitable when the underlying asset depreciates in value
- Moneyness:
  - In-the-money option: positive payoff if exercised immediately
  - At-the-money option: zero payoff if exercised immediately
  - Out-of-the money option: negative payoff if exercised immediately

Option and forward positions

A summary



Options to Complete the Market

Stock's payoff: $x^j = (1, 2, \ldots, S)$ (= state space)

Introduce call options with final payoff at T:

$$C_T = max\{S_T - E, 0\} = [S_T - E]^+$$

$$c_{E=1} = (0, 1, 2, \ldots, S-2, S-1)$$
$$c_{E=2} = (0, 0, 1, \ldots, S-3, S-2)$$
$$\ldots$$
$$c_{E=S-1} = (0, 0, 0, \ldots, 0, 1)$$

Options to Complete the Market

Together with the primitive asset we obtain

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & S-1 & S \\ 0 & 1 & 2 & \cdots & S-2 & S-1 \\ 0 & 0 & 1 & \cdots & S-3 & S-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Homework: check whether this markets are complete.

Cost of Portfolio and Returns

- Price vector $p \in R^J$ of asset prices
- Cost of portfolio h,

$$p \cdot h := \sum_j p^j h^j$$

- If $p^j \neq 0$ the (gross) return vector of asset j is the vector

$$R^j = \frac{x^j}{p^j}$$

Overview

1. Securities Structure
(AD securities, Redundant securities, completeness, …)
2. Pricing
   - LOOP, No arbitrage and existence of state prices
   - Market completeness and uniqueness of state prices
   - Pricing kernel q^*
   - Three pricing formulas (state prices, SDF, EMM)
   - Recovering state prices from options
3. Optimization and Representative Agent
(Pareto efficiency, Welfare Theorems, …)

Pricing

- State space (evolution of states)
- (Risk) preferences
- Aggregation over different agents
- Security structure – prices of traded securities
- *Problem:*
  - *Difficult to observe risk preferences*
  - *What can we say about* ***existence of state prices*** *without assuming specific utility functions for all agents in the economy*

Vector Notation

- Notation: $y, x \in R^n$
  - $y \geq x \Leftrightarrow y^i \geq x^i$ for each $i=1,\ldots,n$.
  - $y > x \Leftrightarrow y \geq x$ and $y \neq x$.
  - $y >> x \Leftrightarrow y^i > x^i$ for each $i=1,\ldots,n$.
- Inner product
  - $y \cdot x = \sum_i yx$
- Matrix multiplication

Three Forms of No-ARBITRAGE

1. Law of one price (LOOP)
   If $h'X = k'X$ then $p \cdot h = p \cdot k$.
2. No strong arbitrage
   There exists no portfolio h which is a strong arbitrage, that is $h'X \geq 0$ and $p \cdot h < 0$.
3. No arbitrage
   There exists no strong arbitrage
   nor portfolio k with $k' X > 0$ and $p \cdot k \leq 0$.

Three Forms of No-ARBITRAGE

- Law of one price is equivalent to
  every portfolio with zero payoff has zero price.
- No arbitrage $\Rightarrow$ no strong arbitrage
  No strong arbitrage $\Rightarrow$ law of one price

Pricing

- Define for each *z* ∈ <X>,

$$q(z) := \{p \cdot h : z = h'X\}$$

- If LOOP holds *q(z)* is a single-valued and linear functional. (i.e. if h’ and h’ lead to same z, then price has to be the same)
- Conversely, if *q* is a linear functional defined in <X> then the law of one price holds.

Pricing

- LOOP $\Rightarrow q(h'X) = p \cdot h$
- A linear functional Q in R^S is a valuation function if $Q(z) = q(z)$ for each z $\in$ <X>.
- Q(z) = q $\cdot$ z for some $q \in R^S$, where $q^s = Q(e_s)$, and e_s is the vector with $e_s^s = 1$ and $e_s^i = 0$ if $i \neq s$
  - e_s is an Arrow-Debreu security
- q is a vector of state prices

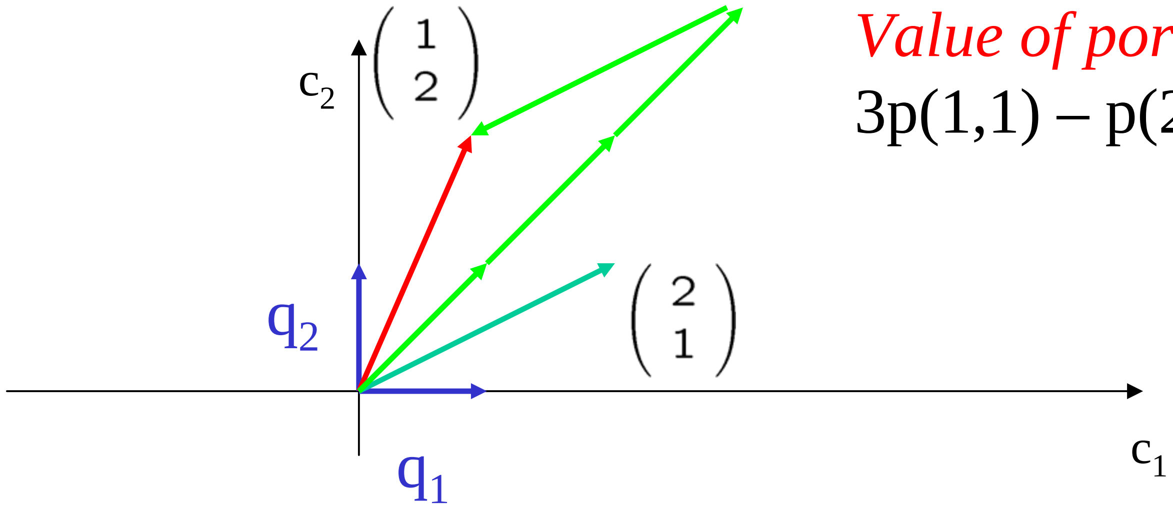
State prices q

- q is a vector of state prices if $p = X q$, that is $p^j = x^j \cdot q$ for each $j = 1,\ldots,J$
- If $Q(z) = q \cdot z$ is a valuation functional then q is a vector of state prices
- Suppose q is a vector of state prices and LOOP holds. Then if $z = h'X$ LOOP implies that

$$q(z) = \sum_j h^j p^j = \sum_j (\sum_s x_s^j q_s) h^j =$$
$$= \sum_s (\sum_j x_s^j h^j) q_s = q \cdot z$$

- $Q(z) = q \cdot z$ is a valuation functional $\Leftrightarrow$ q is a vector of state prices and LOOP holds

State prices q



$p(1,1) = q_1 + q_2$

$p(2,1) = 2q_1 + q_2$

Value of portfolio (1,2)

$3p(1,1) – p(2,1) = 3q_1 + 3q_2 - 2q_1 - q_2$

$= q_1 + 2q_2$

The Fundamental Theorem of Finance

- **Proposition 1.** Security prices exclude arbitrage if and only if there exists a valuation functional with $q >> 0$.
- **Proposition 1'.** Let X be an J � S matrix, and $p \in R^J$. There is no h in R^J satisfying $h \cdot p \leq 0$, $h' X \geq 0$ and at least one strict inequality if, and only if, there exists a vector $q \in R^S$ with $q >> 0$ and $p = X q$.

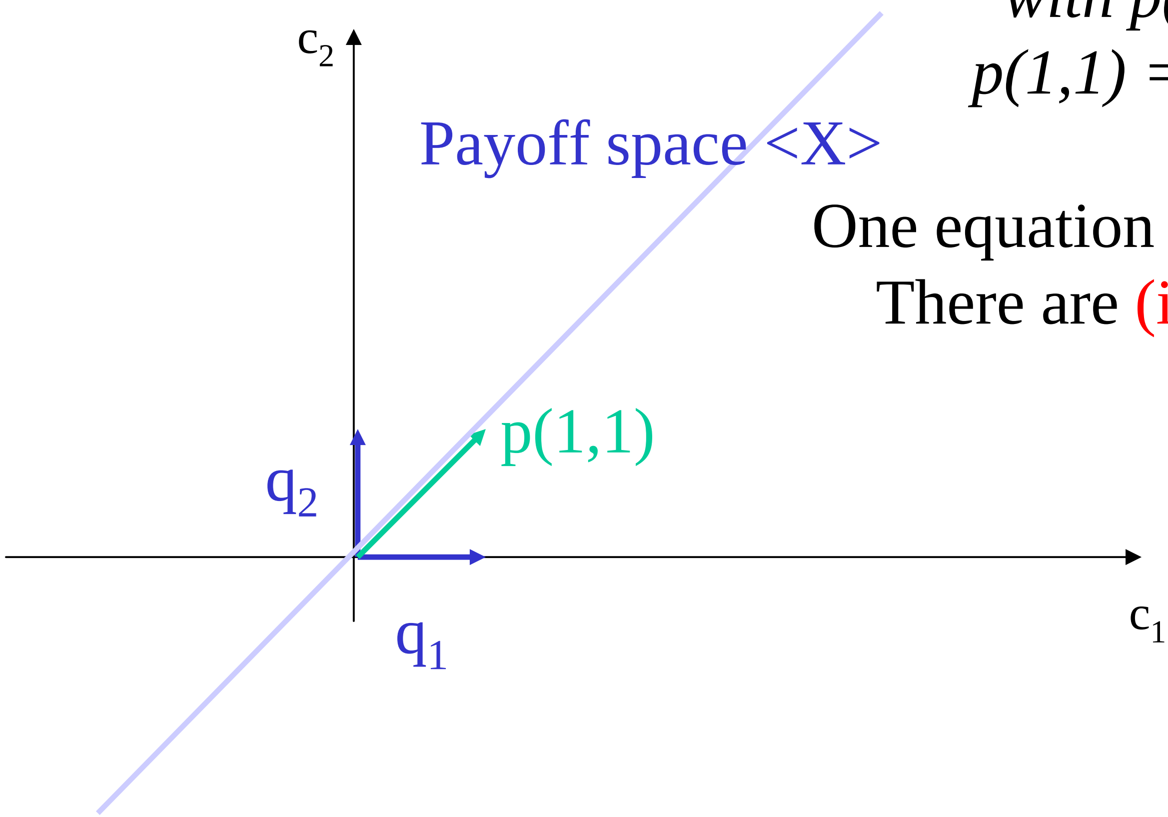
No arbitrage $\Leftrightarrow$ positive state prices

Multiple State Prices q & Incomplete Markets

bond (*1*,*1*) only

What state prices are consistent with p(1,1)?

$$p(1,1) = q_1 + q_2$$



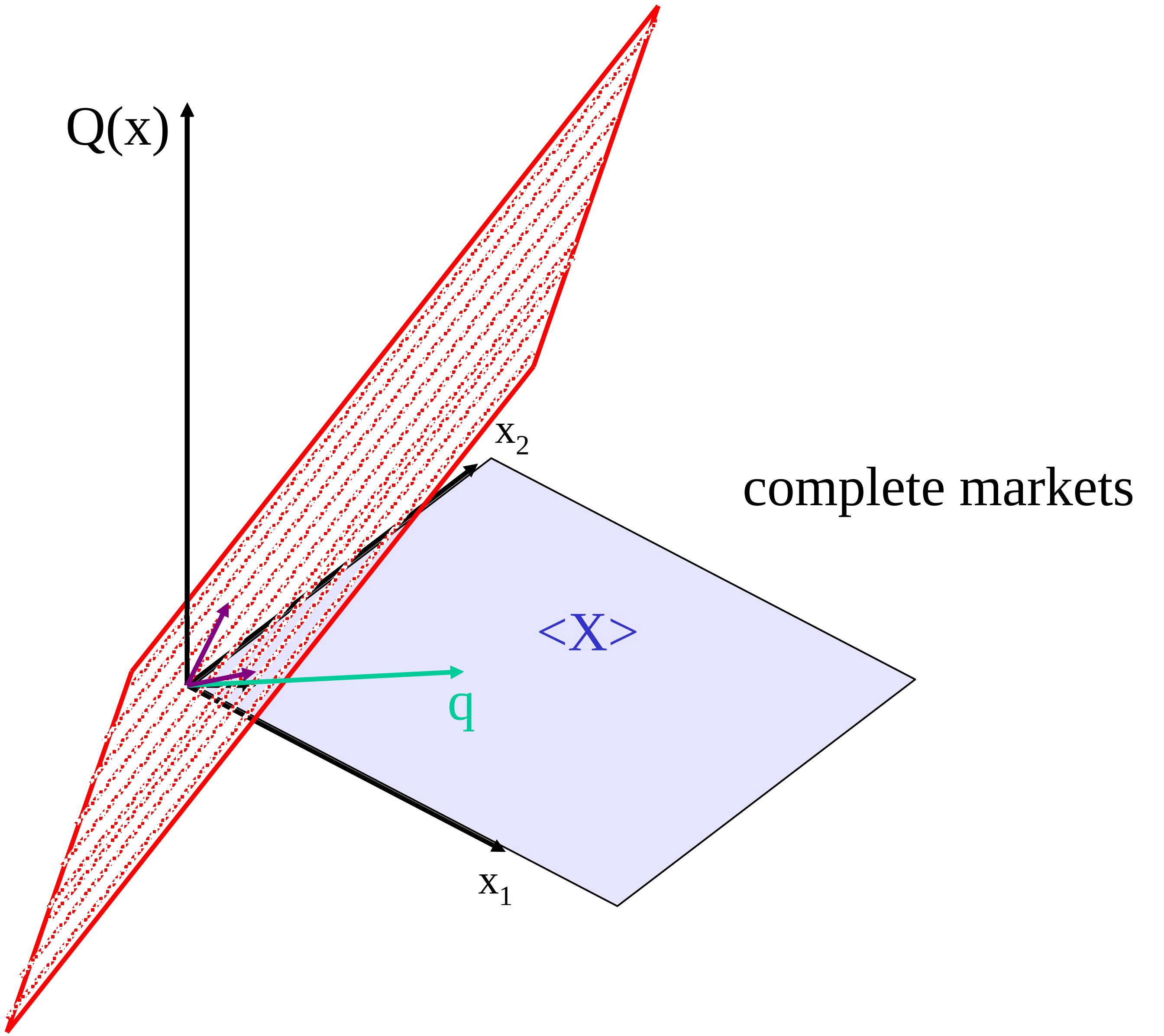
One equation – two unknowns q_1, q_2

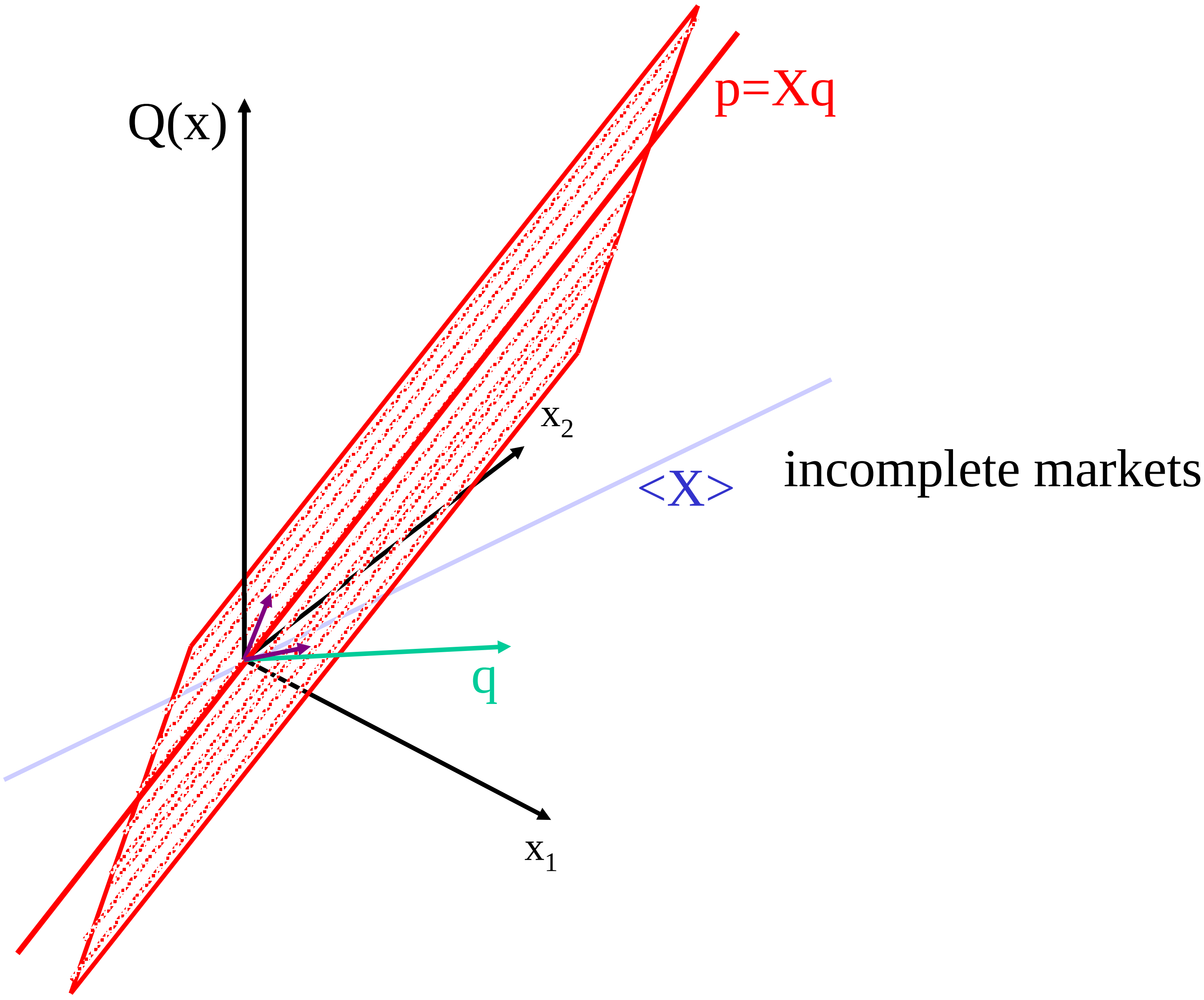
There are (infinitely) many.

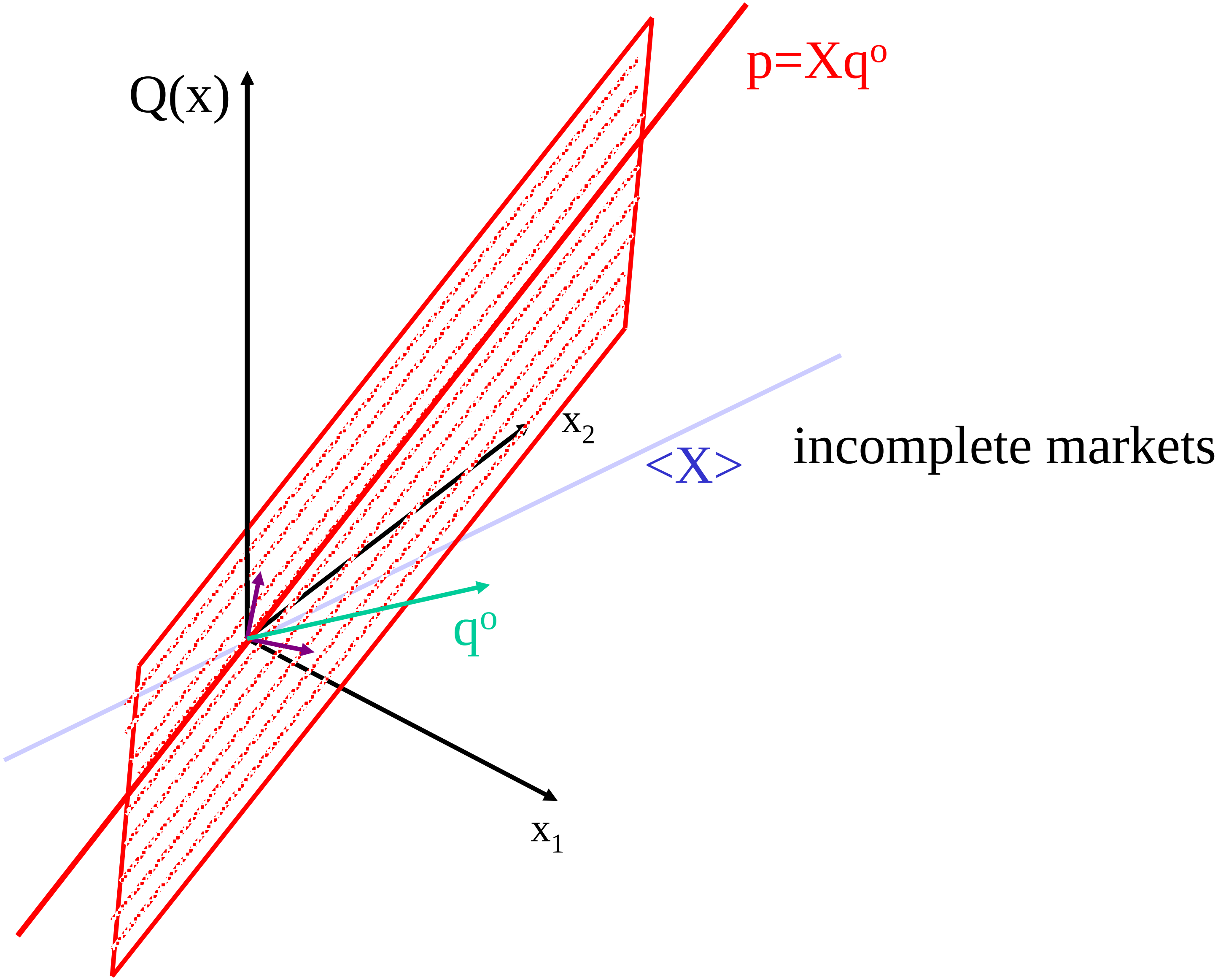
e.g. *if p(1,1)=.9*

$q_1 = .45$, $q_2 = .45$

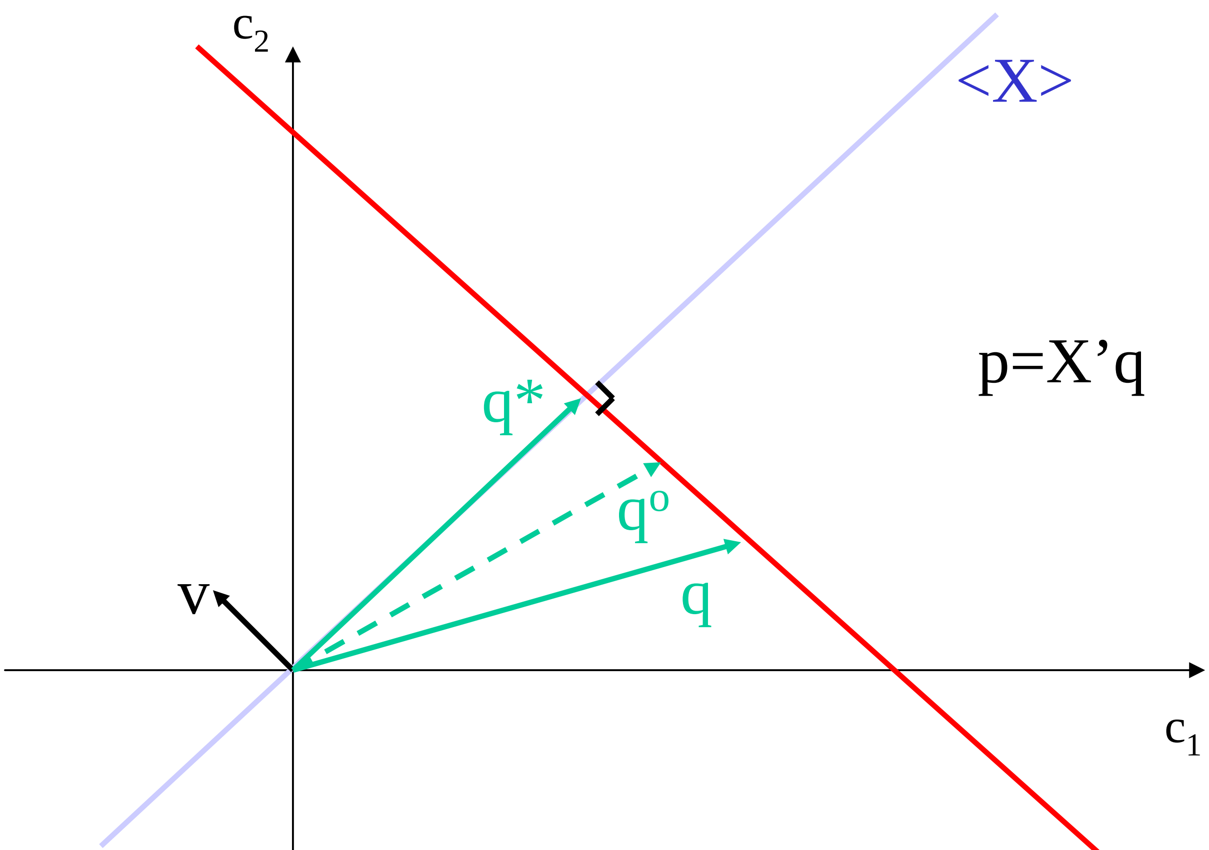
or $q_1 = .35$, $q_2 = .55$


Q(x)
x2
x1
q
<X>
complete markets


p=Xq
Q(x)
x2
incomplete markets
<X>
q
x1


Q(x)
p=Xq°
x_2
incomplete markets
<X>
q°
x_1

Multiple q in incomplete markets



Many possible state price vectors s.t. *p=X'q*.

One is special: q* - it can be replicated as a portfolio.

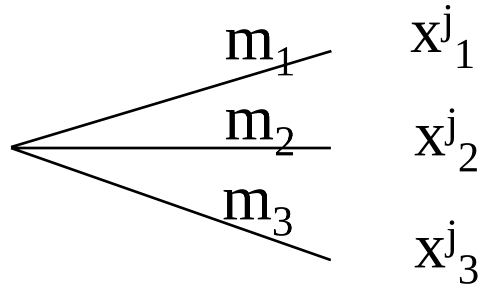
Uniqueness and Completeness

- **Proposition 2.** If markets are complete, under no arbitrage there exists a *unique* valuation functional.

- If markets are not complete, then there exists $v \in R^S$ with $0 = Xv$.
  Suppose there is no arbitrage and let $q >> 0$ be a vector of state prices. Then $q + \alpha v >> 0$ provided α is small enough, and $p = X(q + \alpha v)$. Hence, there are an infinite number of strictly positive state prices.

The Three Asset Pricing Formulas

- State prices $p^j = \sum_s q_s x_s^j$
- Stochastic discount factor $p^j = E[m x^j]$



- Martingale measure $p^j = 1/(1+r^f) E_{\hat{\pi}} [x^j]$

  (reflect risk aversion by
  over(under)weighing the “bad(good)” states!)

Stochastic Discount Factor

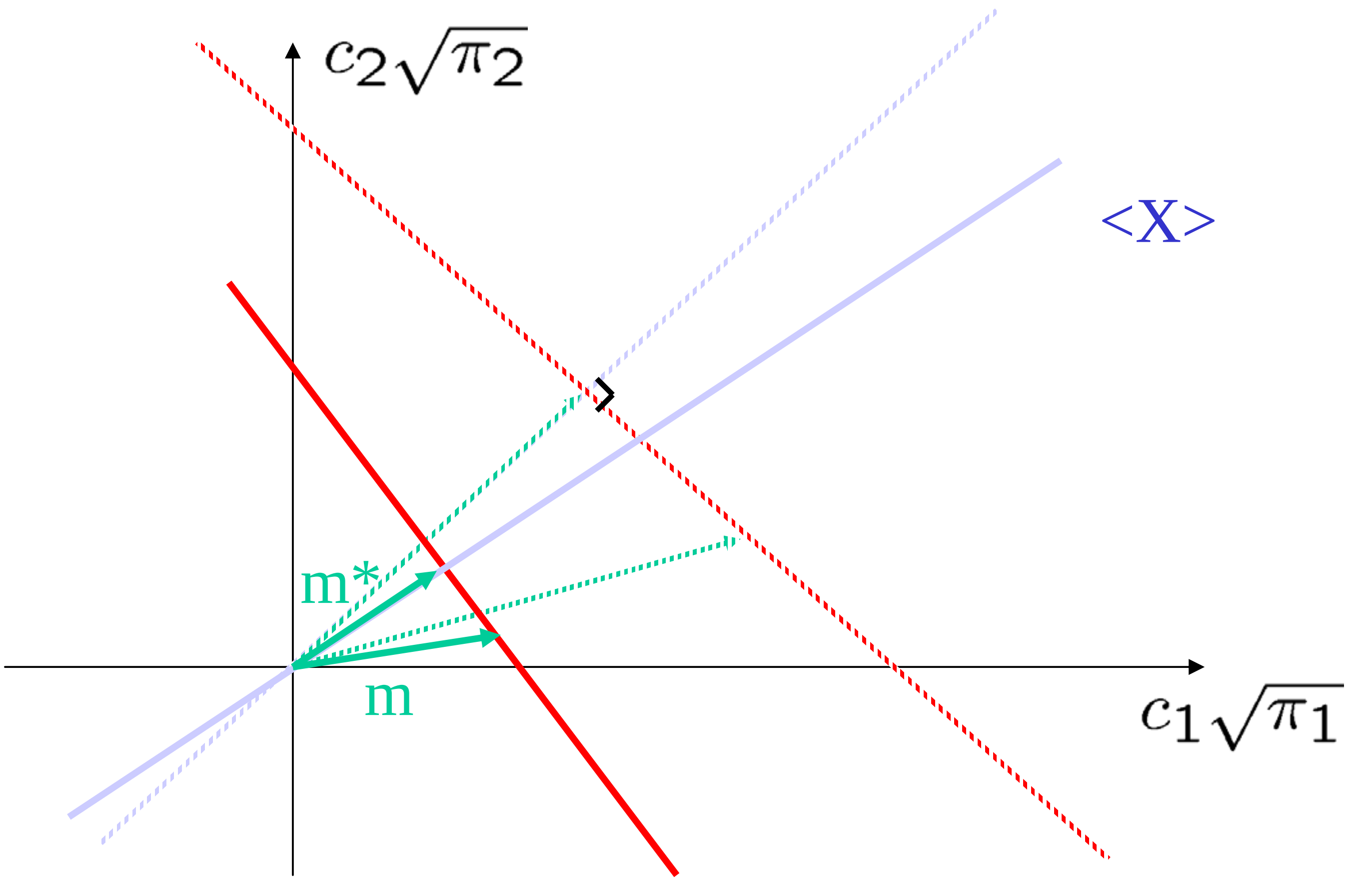
$$p^j = \sum_s q_s x_s^j = \sum_s \pi_s \underbrace{\frac{q_s}{\pi_s}}_{m_s} x_s^j$$

- That is, stochastic discount factor $m_s = q_s/\pi_s$ for all s.

$$p^j = E[mx^j]$$

Stochastic Discount Factor

shrink axes by factor $\sqrt{\pi_s}$



Equivalent Martingale Measure

- Price of any asset $p^j = \sum_s q_s x_s^j$
- Price of a bond $p^{\text{bond}} = \sum_s q_s = \frac{1}{1+r^f}$

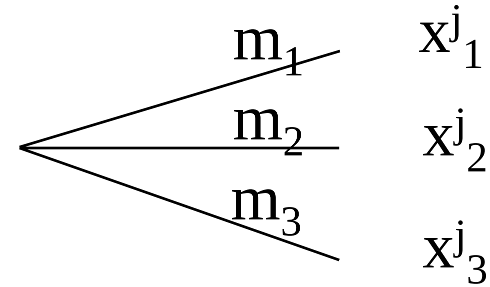
$$p^j = \sum_{s'} q_{s'} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j$$

$$p^j = \frac{1}{1+r^f} \sum_s \underbrace{\frac{q_s}{\sum_{s'} q_{s'}}}_{:=\widehat{\pi}_s} x_s^j$$

$$p^j = \frac{1}{1+r^f} E_{\widehat{\pi}}[x^j]$$

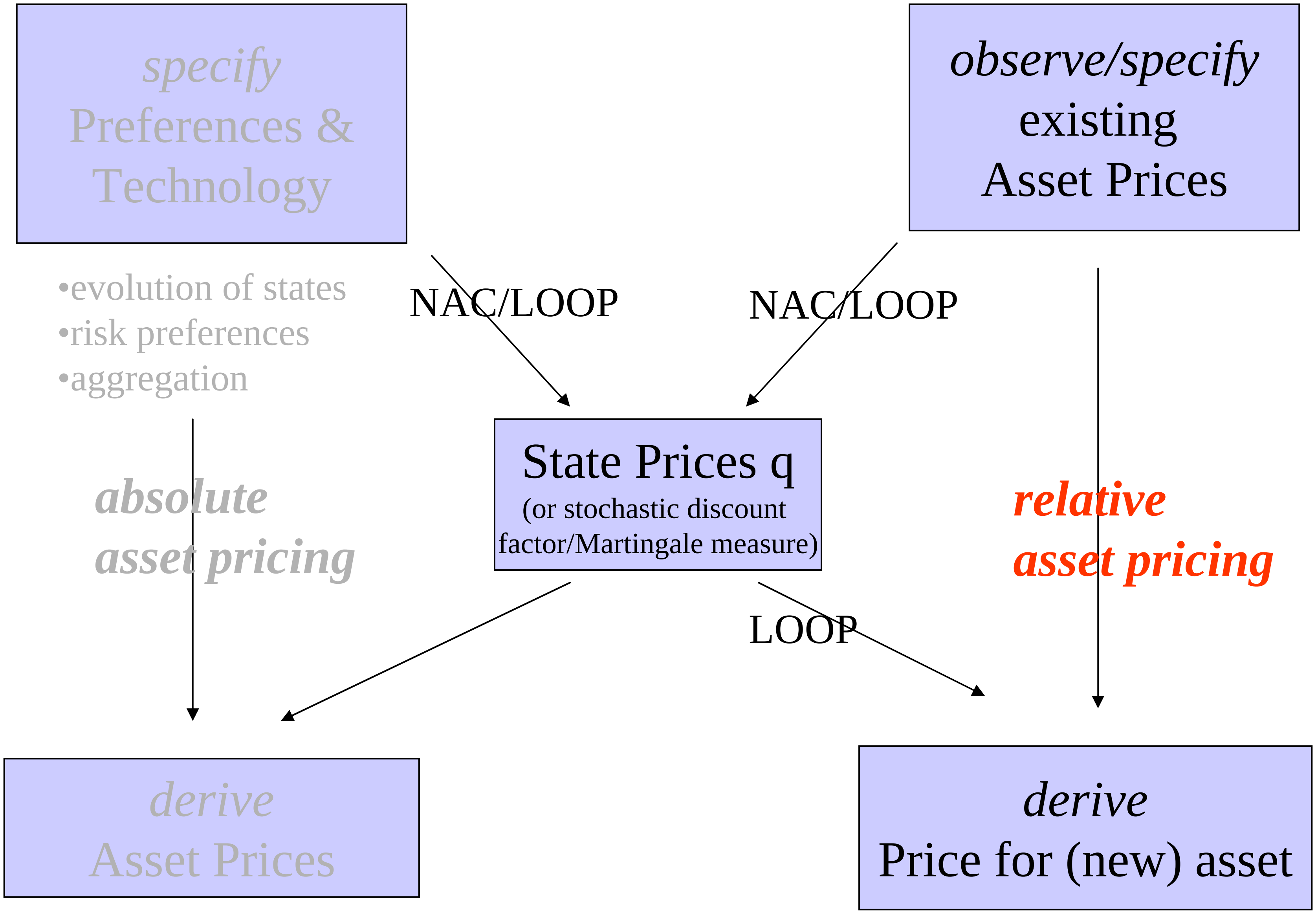
The Three Asset Pricing Formulas

- State prices $p^j = \sum_s q_s \, x_s^j$
- Stochastic discount factor $p^j = E[m x^j]$



- Martingale measure $p^j = 1/(1+r^f) \, E_{\hat{\pi}} [x^j]$

(reflect risk aversion by
over(under)weighing the “bad(good)” states!)



Only works as long as market completeness doesn't change

Recovering State Prices from Option Prices

- Suppose that S_T, the price of the underlying portfolio (we may think of it as a proxy for price of “market portfolio”), assumes a "continuum" of possible values.
- Suppose there are a “continuum” of call options with different strike/exercise prices $\Rightarrow$ markets are complete
- Let us construct the following portfolio:
  for some small positive number ε>0,
  - ❑ Buy one call with $E = \hat{S}_T - \frac{\delta}{2} - \varepsilon$
  - ❑ Sell one call with $E = \hat{S}_T - \frac{\delta}{2}$
  - ❑ Sell one call with $E = \hat{S}_T + \frac{\delta}{2}$
  - ❑ Buy one call with $E = \hat{S}_T + \frac{\delta}{2} + \varepsilon$

Recovering State Prices … (ctd.)

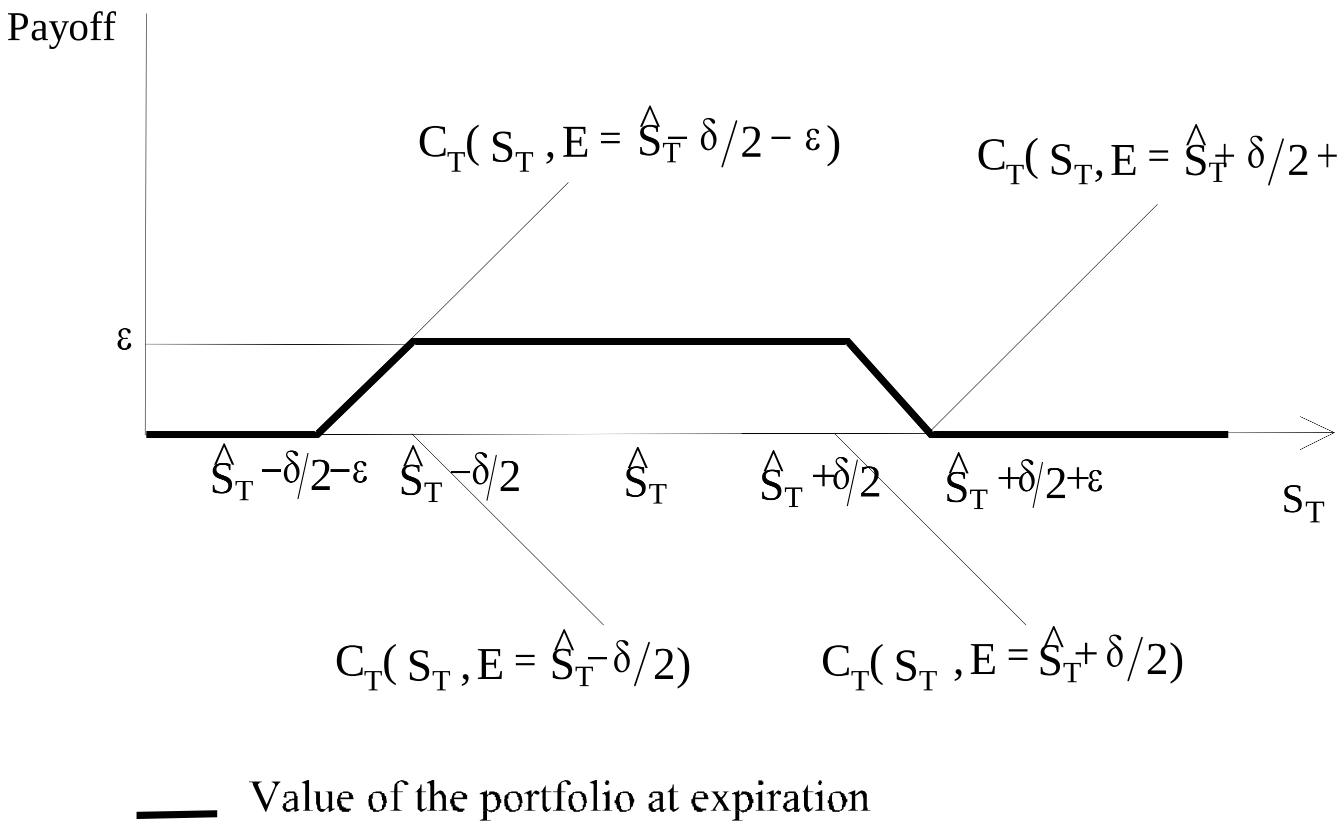


Figure 8-2 Payoff Diagram: Portfolio of Options

Recovering State Prices … (ctd.)

- Let us thus consider buying $^{1}/_{\varepsilon}$ units of the portfolio. The total payment, when $\hat{S}_T - \frac{\delta}{2} \le S_T \le \hat{S}_T + \frac{\delta}{2}$, is $\varepsilon \cdot \frac{1}{\varepsilon} \equiv 1$, for any choice of ε. We want to let $\varepsilon \mapsto 0$, so as to eliminate the payments in the ranges $S_T \in (\hat{S}_T - \frac{\delta}{2} - \varepsilon, \hat{S}_T - \frac{\delta}{2})$ and $S_T \in (\hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \varepsilon)$ .The value of $^{1}/_{\varepsilon}$ units of this portfolio is :

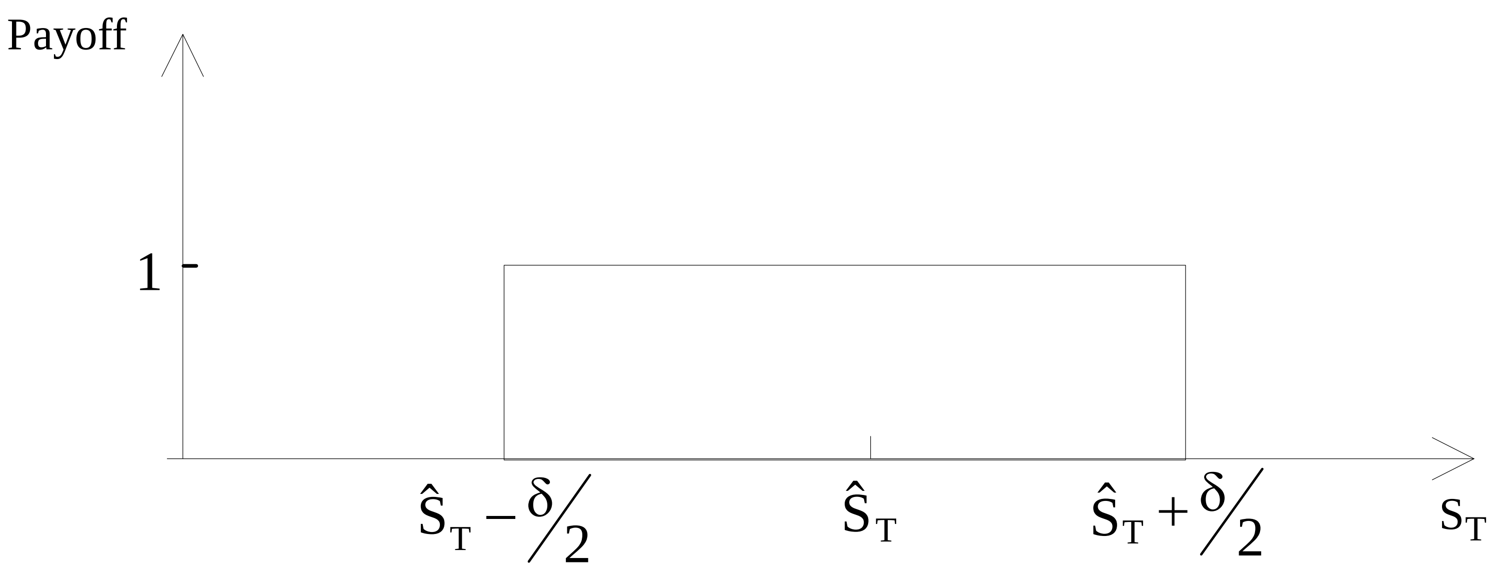
$$\frac{1}{\varepsilon}\left\{C\left(S, E = \hat{S}_T - \tfrac{\delta}{2} - \varepsilon\right) - C\left(S, E = \hat{S}_T - \tfrac{\delta}{2}\right) - \left[C\left(S, E = \hat{S}_T + \tfrac{\delta}{2}\right) - C\left(S, E = \hat{S}_T + \tfrac{\delta}{2} + \varepsilon\right)\right]\right\}$$

Taking the limit $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \mapsto 0} \tfrac{1}{\varepsilon}\left\{C\left(S, E=\hat{S}_T-\tfrac{\delta}{2}-\varepsilon\right)-C\left(S, E=\hat{S}_T-\tfrac{\delta}{2}\right)-\left[C\left(S, E=\hat{S}_T+\tfrac{\delta}{2}\right)-C\left(S, E=\hat{S}_T+\tfrac{\delta}{2}+\varepsilon\right)\right]\right\}$$

$$=-\lim_{\varepsilon \mapsto 0}\underbrace{\left\{\frac{C\left(S, E=\hat{S}_T-\frac{\delta}{2}-\varepsilon\right)-C\left(S, E=\hat{S}_T-\frac{\delta}{2}\right)}{-\varepsilon}\right\}}_{\leq 0}+\lim_{\varepsilon \mapsto 0}\underbrace{\left\{\frac{C\left(S, E=\hat{S}_T+\frac{\delta}{2}+\varepsilon\right)-C\left(S, E=\hat{S}_T+\frac{\delta}{2}\right)}{\varepsilon}\right\}}_{\leq 0}$$

$$=-\frac{\partial C}{\partial E}\left(S, E=\hat{S}_T-\frac{\delta}{2}\right)+\frac{\partial C}{\partial E}\left(S, E=\hat{S}_T+\frac{\delta}{2}\right)$$



Divide by δ and let $\delta \rightarrow 0$ to obtain state price **density** as $\partial^2 C/\partial E^2$.

Recovering State Prices … (ctd.)

Evaluating following cash flow

$$\tilde{CF}_T = \begin{cases} 0 \text{ if } S_T & \notin \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right] \\ 50000 \text{ if } S_T & \in \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right] \end{cases}.$$

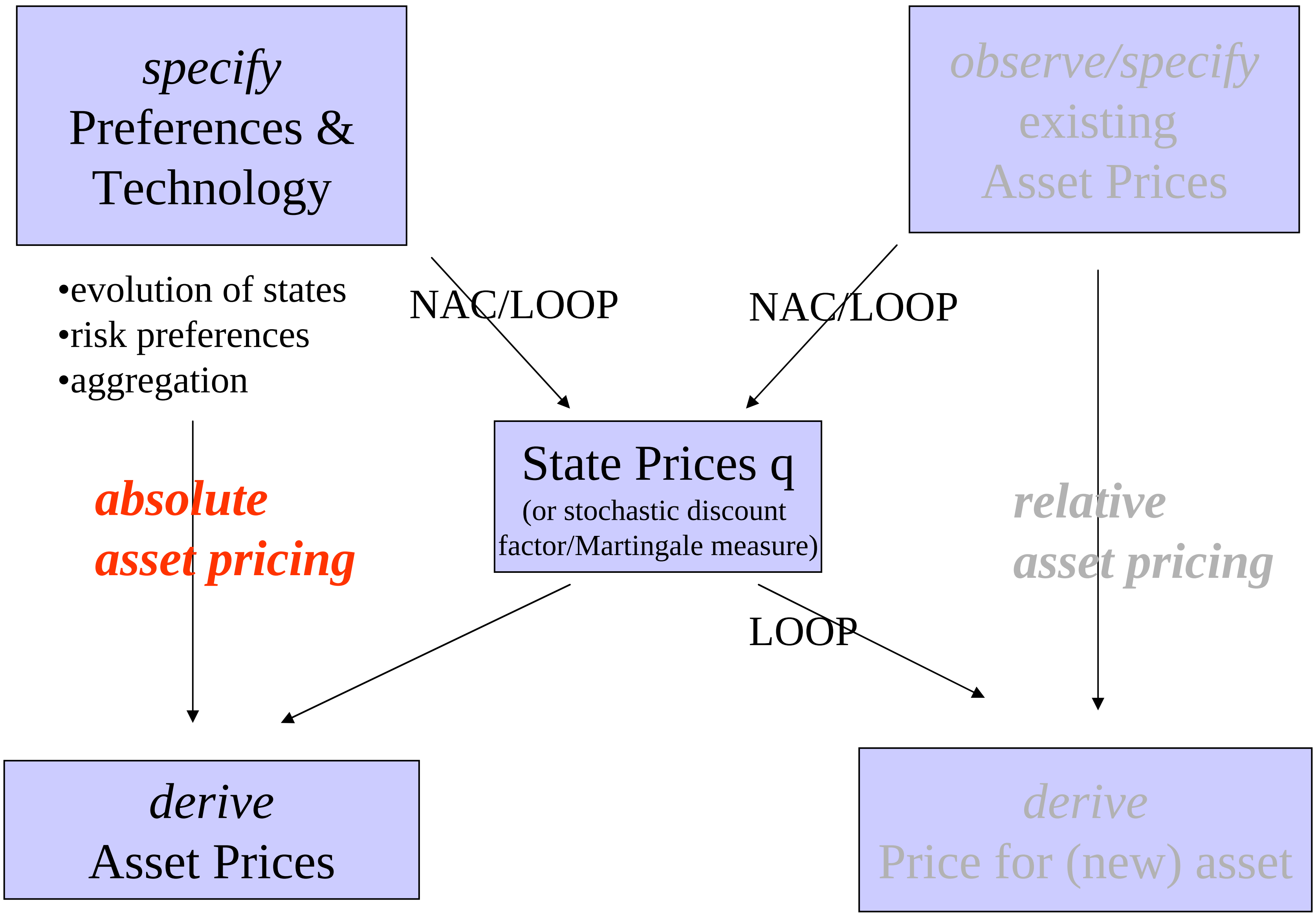
The value today of this cash flow is :

$$50000[\frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2}) - \frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2})]$$

$$q(S_T^1, S_T^2) = \frac{\partial C}{\partial E}(S, E = S_T^2) - \frac{\partial C}{\partial E}(S, E = S_T^1)$$

Table 8.1 Pricing an Arrow-Debreu State Claim

E	C(S,E)	Cost of position	Payoff if S_T =							ΔC	Δ(ΔC)= q_s
			7	8	9	10	11	12	13		
7	3.354										
										-0.895	
8	2.459										0.106
										-0.789	
9	1.670	+1.670	0	0	0	1	2	3	4		0.164
										-0.625	
10	1.045	-2.090	0	0	0	0	-2	-4	-6		**0.184**
										-0.441	
11	0.604	+0.604	0	0	0	0	0	1	2		0.162
										-0.279	
12	0.325										0.118
										-0.161	
13	0.164										
		0.184	0	0	0	1	0	0	0		



Only works as long as market completeness doesn't change

Overview

1. Securities Structure
   (AD securities, Redundant securities, completeness, …)
2. Pricing (no arbitrage, state prices, SDF, EMM …)
3. Optimization and Representative Agent
   - Marginal Rate of Substitution (MRS)
   - Pareto Efficiency
   - Welfare Theorems
   - Representative Agent Economy

Representation of Preferences

A preference ordering is (i) complete, (ii) transitive, (iii) continuous [and (iv) relatively stable] can be represented by a utility function, i.e.

$$(c_0,c_1,\ldots,c_S) \succ (c'_0,c'_1,\ldots,c'_S)$$
$$\Leftrightarrow \; U(c_0,c_1,\ldots,c_S) > U(c'_0,c'_1,\ldots,c'_S)$$

(more on risk preferences in next lecture)

Agent's Optimization

- Consumption vector $(c_0, c_1) \in R_+ \times R_+^S$
- Agent *i* has $U^i : R_+ \times R_+^S \to R$
  endowments $(e_0, e_1) \in R_+ \times R_+^S$
- U^i is quasiconcave $\{c: U^i(c) \geq v\}$ is convex for each real v
  - U^i is concave: for each $0 \geq \alpha \geq 1$,
    $U^i(\alpha c + (1-\alpha)c') \geq \alpha U^i(c) + (1-\alpha) U^i(c')$
- $\partial U^i / \partial c_0 > 0$, $\partial U^i / \partial c_1 >> 0$

Agent's Optimization

- Portfolio consumption problem

$$\max_{c_0, c_1, h} U^i(c_0, c_1)$$
$$\text{subject to } (i) \quad 0 \le c_0 \le e_0 - p \cdot h$$
$$\text{and } (ii) \quad 0 \le c_1 \le e_1 + X'h$$

$$U^i(c_0, \vec{c}_1) - \lambda[c_0 - e_0 + ph] - \vec{\mu}[\vec{c}_1 - \vec{e}_1 - h'X]$$

$$\begin{aligned} \text{FOC} \quad c_0 : \quad & \frac{\partial U^i}{\partial c_0}(c^*) &=& \quad \lambda \\ c_s : \quad & \frac{\partial U^i}{\partial c_s}(c^*) &=& \quad \mu_s \\ h : \quad & \lambda \vec{p} &=& \quad X \vec{\mu} \\ \Leftrightarrow \quad & p^j &=& \quad \textstyle\sum_s \frac{\mu_s}{\lambda} x_s^j \end{aligned}$$

Agent's Optimization

$$p^j = \sum_s \frac{\partial U^i / \partial c_s}{\partial U^i / \partial c_0} x_s^j$$

For time separable utility function

$$U^i(c_0, \overrightarrow{c}) = u(c_0) + \delta u(\overrightarrow{c})$$

and expected utility function (later more)

$$U^i(c_0, \overrightarrow{c_1}) = u(c_0) + \delta E[u(c)]$$

$$p^j = \sum_s \pi_s \delta \frac{\partial u^i / \partial c_s}{\partial u^i / \partial c_0} x_s^j$$

Welfare Theorems

- *First Welfare Theorem*. If markets are complete, then the equilibrium allocation is Pareto optimal.
  - State price is unique q. All $MRS^i(c^*)$ coincide with unique state price q.
- *Second Welfare Theorem*. Any Pareto efficient allocation can be decentralized as a competitive equilibrium.

Representative Agent & Complete Markets

- Aggregation Theorem 1: Suppose
  - ❑markets are complete

Then asset prices in economy with *many agents* are identical to an economy with a *single agent/planner* whose utility is

$$U(c) = \sum_k \alpha_k u^k(c),$$

where α^k is the welfare weights of agent k.

and the single agent consumes the aggregate endowment.

Representative Agent & HARA utility world

- Aggregation Theorem 2: Suppose
  - riskless annuity and endowments are tradable.
  - agents have common beliefs
  - agents have a common rate of time preference
  - agents have LRT (HARA) preferences with

$$R_A(c) = 1/(A_i + Bc) \Rightarrow \text{linear risk sharing rule}$$

Then asset prices in economy with *many agents* are identical to a *single agent* economy with HARA preferences with $R_A(c) = 1/(\sum_i A_i + B)$.

Overview

1. Securities Structure
   (AD securities, Redundant securities, completeness, …)
2. Pricing (no arbitrage, state prices, SDF, EMM …)
3. Optimization and Representative Agent
   (Pareto efficiency, Welfare Theorems, …)

Extra Material

Follows!

Portfolio restrictions

- Suppose that there is a short-sale restriction

$$h \in \mathcal{C} = \{h : h^j \geq -b^j, j \in \mathcal{J}_0\}$$

where $b \geq 0, \mathcal{J}_0 \subset \{1, \ldots, J\}$

- $\mathcal{C}$ is a convex set
- $< \tilde{X} >= \{z \in R^S : z = h'X' \text{for some } h \in \mathcal{C}\}$
- $< \tilde{X} >$ is a convex set
- F or z $\in < \tilde{X} >$ let (cheapest portfolio replicating z)

$$\tilde{q}(z) = \inf_h \{p \cdot h : z = h'X', h \in \mathcal{C}\}$$

Restricted/Limited Arbitrage

- An arbitrage is limited if it involves a short position in a security $j \in \mathcal{J}_0$
- In the presence of short-sale restrictions, security prices exclude (unlimited) arbitrage (payoff ∞) if, and only if, here exists a *q >> 0* such that

$$p^j \geq x^j \cdot q \quad \forall j \in \mathcal{J}_0$$

$$p^j = x^j \cdot q \quad \forall j \notin \mathcal{J}_0$$

- Intuition: *q = MRS^i* from optimization problem

some agents wished they could short-sell asset

Portfolio restrictions (ctd.)

- As before, we may define $R^f = 1 / \sum_s q_s$, and $\widehat{\pi}_s$ can be interpreted as risk-neutral probabilities
- $R^f p^j \geq E^\pi [x^j]$, with = if $j \notin \mathcal{J}_0$
- $1/R^f$ is the price of a risk-free security that is not subject to short-sale constraint.

Portfolio restrictions (ctd.)

- Portfolio consumption problem

  $\max_{c_0, c_1, h} U^i(c_0, c_1)$

  subject to (i) $\quad 0 \le c_0 \le w_0 - p \cdot h$

  and (ii) $\quad 0 \le c_1 \le w_1 + h'X'$ and $h \in \mathcal{J}_0$

- Proposition 4: Suppose c*>>0 solves problem s.t. $h^j \ge -b^j$ for $j \in \mathcal{J}_0$. Then there exists positive real numbers $\lambda, \mu_1, \mu_2, \ldots, \mu_S$, such that
  - $\partial U^i / \partial c_0 (c^*) = \lambda$
  - $\partial U^i / \partial c_0 (c^*) = (\mu_1, \ldots, \mu_S)$
  - $p^j \ge \sum_s \frac{\mu_s}{\lambda} x_s^j = \sum_s MRS_{0,s}^i x_s^j$
  - $p^j = \sum_s \frac{\mu_s}{\lambda} x_s^j$, if $\notin \mathcal{J}_0$ or $h^j > -b^j$

The converse is also true.

FOR LATER USE Stochastic Discount Factor

$$p^j = \sum_s \underbrace{\pi_s \, \delta \underbrace{\frac{\partial u^i(c^*)/\partial c_s}{\partial u^i(c^*)/\partial c_0}}_{m_s}}_{q_s} x_s^i$$

- That is, stochastic discount factor $m_s = q_s/\pi_s$ for all s.

$$p^j = \sum_s \pi_s m_s x_s^j$$
$$p^j = E[mx^j]$$