Lecture 02: Risk Preferences and Savings/Portfolio Choice

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State-by-state Dominance

- State-by-state dominance ⇒ incomplete ranking
- « riskier »

Table 2.1 Asset Payoffs ($)

	t = 0	t = 1	
	Cost at t=0	Value at t=1 $\pi_1 = \pi_2 = ½$	
		s = 1	s = 2
investment 1	- 1000	1050	1200
investment 2	- 1000	500	1600
investment 3	- 1000	1050	1600

- investment 3 state by state dominates 1.

State-by-state Dominance (ctd.)

Table 2.2 State Contingent ROR (r)

	State Contingent ROR (r)			
	s = 1	s = 2	Er	σ
Investment 1	5%	20%	12.5%	7.5%
Investment 2	-50%	60%	5%	55%
Investment 3	5%	60%	32.5%	27.5%

- Investment 1 mean-variance dominates 2
- BUT investment 3 does not m-v dominate 1!

State-by-state Dominance (ctd.)

Table 2.3 State Contingent Rates of Return

	State Contingent Rates of Return	
	s = 1	s = 2
investment 4	3%	5%
investment 5	3%	8%
	$\pi_1 = \pi_2 = ½$ $E[r_4] = 4\%; \quad \sigma_4 = 1\%$ $E[r_5] = 5.5\%; \sigma_5 = 2.5\%$	

- What is the trade-off between risk and expected return?
- Investment 4 has a higher Sharpe ratio $(E[r]-r^f)/\sigma$ than investment 5 for $r^f = 0$.

Stochastic Dominance

- Stochastic dominance can be defined independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. (“risk-preference-free”)
- Less “demanding” than state-by-state dominance

Stochastic Dominance

- Still incomplete ordering
  - “More complete” than state-by-state ordering
  - State-by-state dominance ⇒ stochastic dominance
  - Risk preference not needed for ranking!
    - independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. (“risk-preference-free”)
- Next Section:
  - Complete preference ordering and utility representations

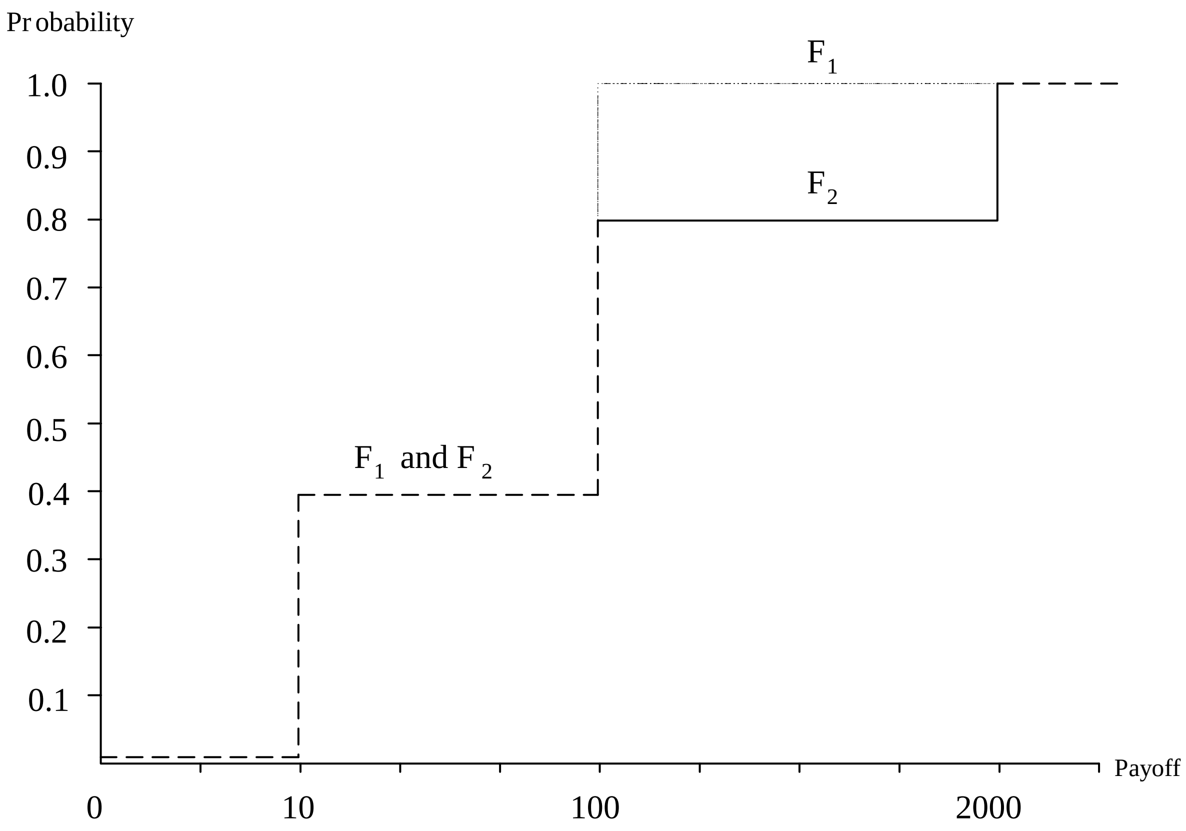
Homework: Provide an example which can be ranked according to FSD , but not according to state dominance.

Table 3-1 Sample Investment Alternatives

States of nature	1	2	3
Payoffs	10	100	2000
Proba Z_1	.4	.6	0
Proba Z_2	.4	.4	.2

$EZ_1 = 64, \ \sigma_{z_1} = 44$

$EZ_2 = 444, \ \sigma_{z_2} = 779$



First Order Stochastic Dominance

- Definition 3.1 : *Let* $F_A(x)$ *and* $F_B(x)$, *respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval [a,b]. We say that* $F_A(x)$ *first order stochastically dominates (FSD)* $F_B(x)$ *if and only if for all* $x \in [a,b]$

$$F_A(x) \leq F_B(x)$$

Homework: Provide an example which can be ranked according to FSD , but not according to state dominance.

First Order Stochastic Dominance

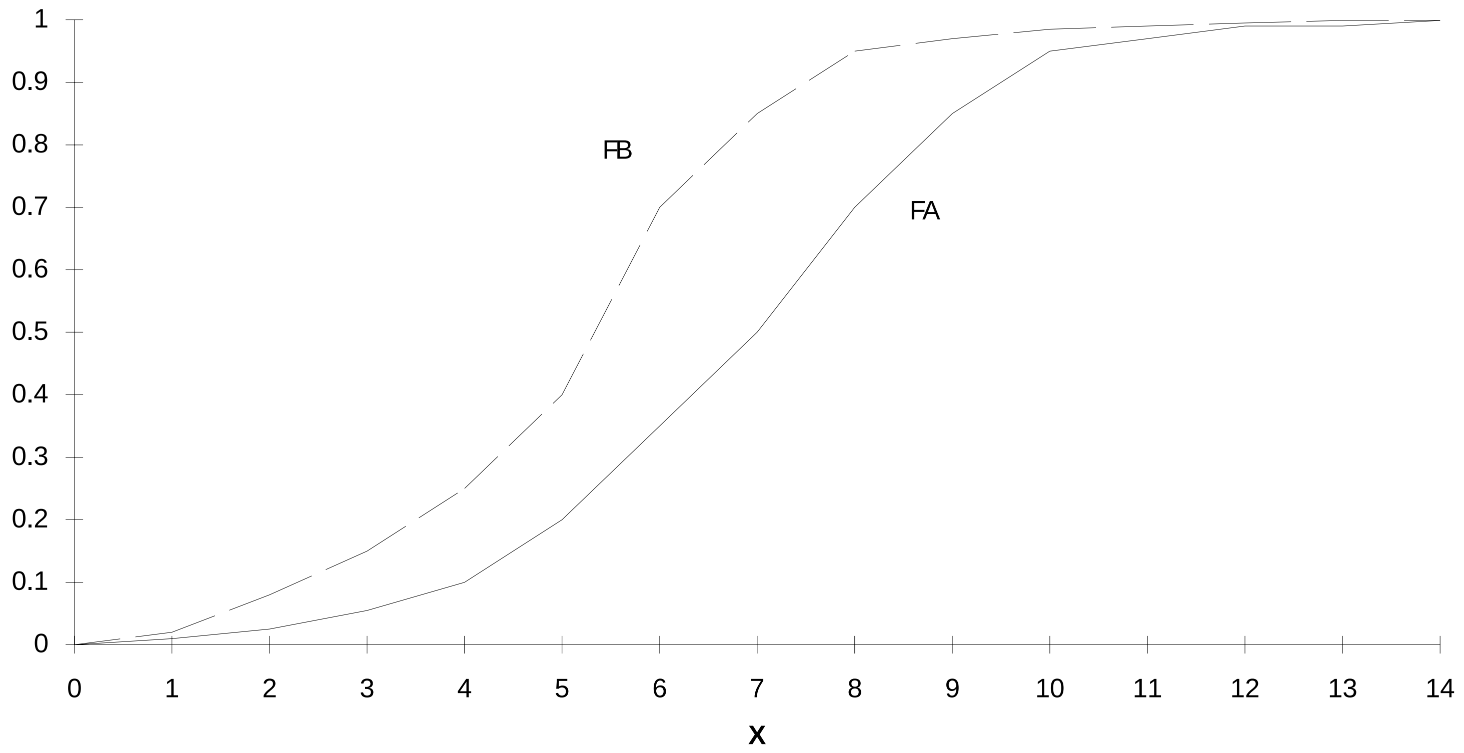


Table 3-2 Two Independent Investments

Investment 3		Investment 4	
Payoff	Prob.	Payoff	Prob.
4	0.25	1	0.33
5	0.50	6	0.33
12	0.25	8	0.33

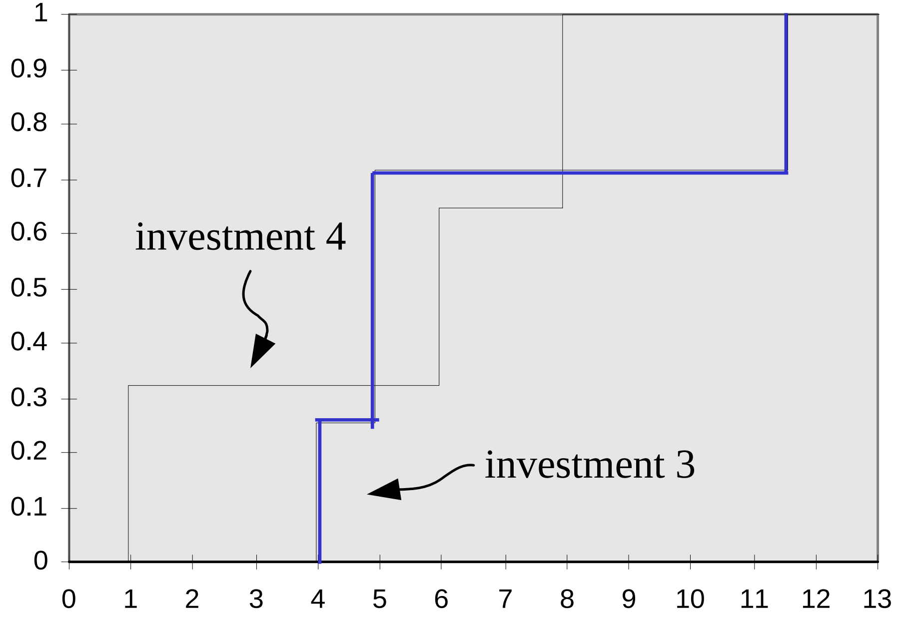


Figure 3-6 Second Order Stochastic Dominance Illustrated

Second Order Stochastic Dominance

- Definition 3.2: *Let* $F_A(\tilde{x})$, $F_B(\tilde{x})$ *, be two cumulative probability distribution for random payoffs in* $[a, b]$. *We say that* $F_A(\tilde{x})$ *second order stochastically dominates (SSD)* $F_B(\tilde{x})$ *if and only if for any* x :

$$\int_{-\infty}^{x} \left[F_B(t) - F_A(t) \right] dt \geq 0$$

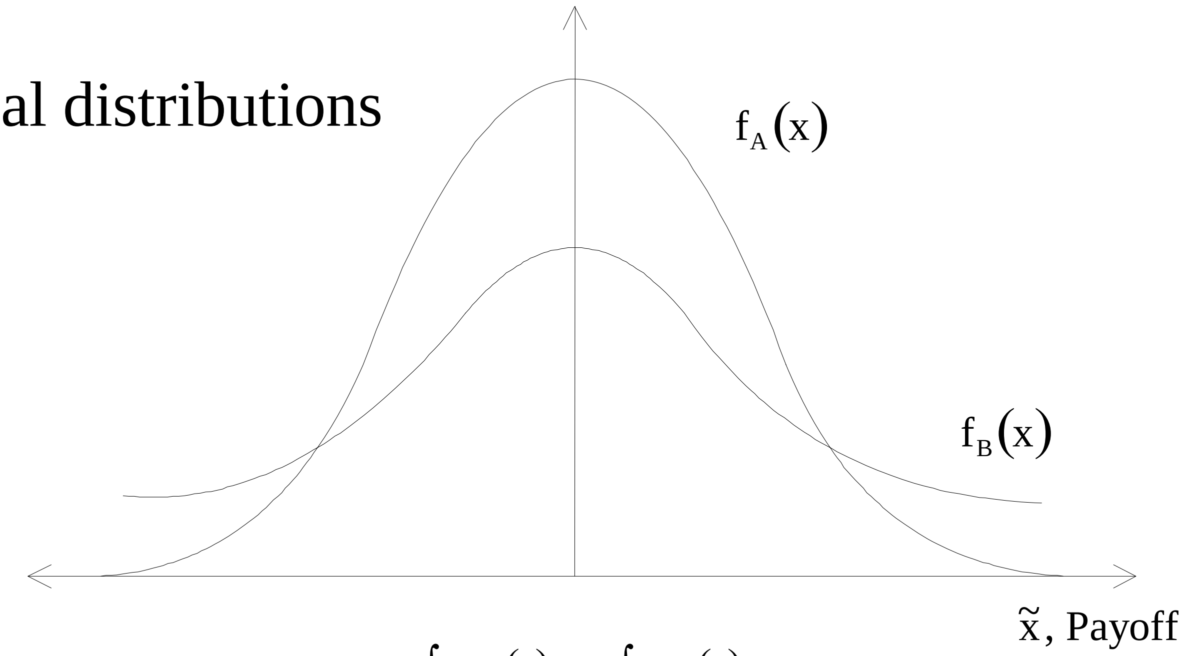
(with strict inequality for some meaningful interval of values of *t*).

Mean Preserving Spread

$$x_B = x_A + z \qquad (3.8)$$

where z is independent of x_A and has zero mean

for normal distributions



$$\mu = \int x\, f_A(x)dx = \int x\, f_B(x)dx$$

Figure 3-7 Mean Preserving Spread

Mean Preserving Spread & SSD

- Theorem 3.4 : *Let* $F_A(\bullet)$ *and* $F_B(\bullet)$ *be two distribution functions defined on the same state space with identical means. Then the follow statements are equivalent :*
  - $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$
  - $F_B(\tilde{x})$ *is a mean preserving spread of* $F_A(\tilde{x})$ *in the sense of Equation* (3.8) *above.*

Expected Utility & Stochastic Dominance

- <u>Theorem 3. 2</u> : *Let* $F_A(\tilde{x})$, $F_B(\tilde{x})$, *be two cumulative probability distribution for random payoffs* $\tilde{x} \in [a, b]$. *Then* $F_A(\tilde{x})$ FSD $F_B(\tilde{x})$ *if and only if for all non decreasing utility functions* $U(\bullet)$.

$$E_A U(\tilde{x}) \geq E_B U(\tilde{x})$$

Expected Utility & Stochastic Dominance

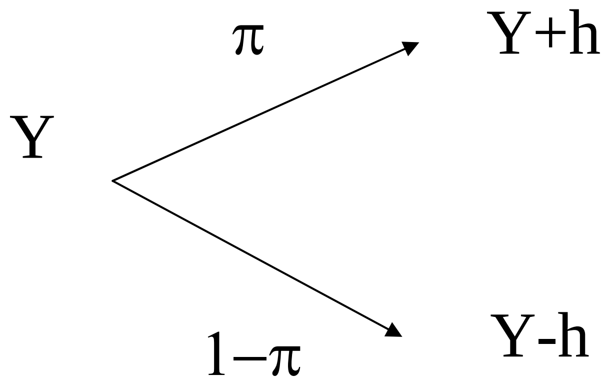
- Theorem 3. 3 : *Let* $F_A(\tilde{x})$, $F_B(\tilde{x})$, *be two cumulative probability distribution*
  for random payoffs $\tilde{x}$ *defined on* $[a, b]$.
  Then, $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$ *if and only if* $E_A U(\tilde{x}) \geq E_B U(\tilde{x})$
  for all non decreasing and concave U.

Arrow-Pratt measures of risk aversion and their interpretations

- absolute risk aversion $= - \dfrac{U''(Y)}{U'(Y)} \equiv R_A(Y)$
- relative risk aversion $= - \dfrac{Y\,U''(Y)}{U'(Y)} \equiv R_R(Y)$
- risk tolerance $= \dfrac{1}{R_A}$

Absolute risk aversion coefficient

$$R_A = -\frac{U''(Y)}{U'(Y)}$$



π

Y+h

Y

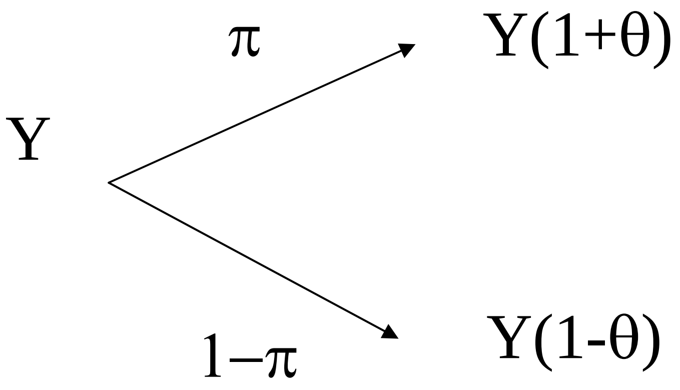
1−π

Y-h

$$\pi(Y,h) = \frac{1}{2} + \frac{1}{4} h R_A(Y) + HOT$$

Relative risk aversion coefficient

$$R_R = -\frac{U''(Y)}{U'(Y)}Y$$



Y

π

Y(1+θ)

1–π

Y(1-θ)

$$\pi(Y,\theta) = \tfrac{1}{2} + \tfrac{1}{4}\theta R_R(Y) + HOT$$

Homework: Derive this result.

CARA and CRRA-utility functions

- Constant Absolute RA utility function

$$U(Y) = -e^{-\rho Y}$$

- Constant Relative RA utility function

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} \text{ for } \gamma \neq 1$$
$$U(Y) = lnY \quad \text{ for } \gamma = 1$$

Investor ’s Level of Relative Risk Aversion

$$\frac{(Y+CE)^{1-\gamma}}{1-\gamma} = \frac{\tfrac{1}{2}(Y+50{,}000)^{1-\gamma}}{1-\gamma} + \frac{\tfrac{1}{2}(Y+100{,}000)^{1-\gamma}}{1-\gamma}$$

	$\gamma = 0$	CE = 75,000 (risk neutrality)
	$\gamma = 1$	CE = 70,711
Y=0	$\gamma = 2$	CE = 66,246
	$\gamma = 5$	CE = 58,566
	$\gamma = 10$	CE = 53,991
	$\gamma = 20$	CE = 51,858
	$\gamma = 30$	CE = 51,209
Y=100,000	$\gamma = 5$	CE = 66,530

Risk aversion and Portfolio Allocation

- No savings decision (consumption occurs only at *t=1*)
- Asset structure
  - One risk free bond with net return r_f
  - One risky asset with random net return r (a =quantity of risky assets)

$$\max_a E[U(Y_0(1+r_f)+a(r-r_f))]$$

FOC:

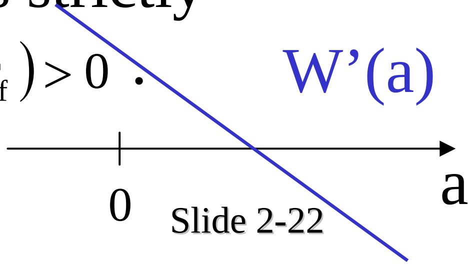
$$E[U'(Y_0(1+r_f)+a(r-r_f))(r-r_f)]=0$$

- Theorem 4.1: Assume $U'(\,) > 0$, and $U''(\,) < 0$ and let $\hat{a}$ denote the solution to above problem. Then

$$\begin{aligned} \hat{a} > 0 &\quad \text{if and only if } E\tilde{r} > r_f \\ \hat{a} = 0 &\quad \text{if and only if } E\tilde{r} = r_f \\ \hat{a} < 0 &\quad \text{if and only if } E\tilde{r} < r_f \quad . \end{aligned}$$

- Define $W(a) = E\{U(Y_0(1+r_f)+a(\tilde{r}-r_f))\}$. The FOC can then be written $W'(a) = E[U'(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)] = 0$. By risk aversion ($U''<0$), $W''(a) = E\left[U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)^2\right]$ < 0, that is, $W'(a)$ is everywhere decreasing. It follows that $\hat{a}$ will be positive if and only if $W'(0) = U'(Y_0(1+r_f))E(\tilde{r}-r_f) > 0$

(since then a will have to be increased from its value of 0 to achieve equality in the FOC). Since U' is always strictly positive, this implies $\hat{a} > 0$ if and only if $E(\tilde{r}-r_f) > 0$.



The other assertion follows similarly. □

Portfolio as wealth changes

- Theorem 4.4 (Arrow, 1971): Let $\hat{a} = \hat{a}(Y_0)$ be the solution to max-problem above; then:

(i) $\frac{\partial R_A}{\partial Y} < 0 \text{ (DARA) implies } \frac{\partial \hat{a}}{\partial Y_0} > 0$

(ii) $\frac{\partial R_A}{\partial Y} = 0 \text{ (CARA) implies } \frac{\partial \hat{a}}{\partial Y_0} = 0$

(iii) $\frac{\partial R_A}{\partial Y} > 0 \text{ (IARA) implies } \frac{\partial \hat{a}}{\partial Y_0} < 0$

Portfolio as wealth changes

- Theorem 4.5 (Arrow 1971): If, for all wealth levels Y,

(i) $$\frac{\partial R_R}{\partial Y} = 0 \text{ (CRRA) implies } \eta = 1$$

(ii) $$\frac{\partial R_R}{\partial Y} < 0 \text{ (DRRA) implies } \eta > 1$$

(iii) $$\frac{\partial R_R}{\partial Y} > 0 \text{ (IRRA) implies } \eta < 1$$

where ♒ = da/a / dY/Y (elasticity)

Log utility & Portfolio Allocation

U(Y) = ln Y.

$$E\{\frac{\tilde{r}-r_f}{Y_0(1+r_f)+a(\tilde{r}-r_f)}\} = 0$$

2 states, where $r_2 > r_f > r_1$

$$\frac{a}{Y_0} = \frac{(1+r_f)[E[\tilde{r}]-r_f]}{-(r_1-r_f)(r_2-r_f)} > 0$$

Constant fraction of wealth is invested in risky asset!

Portfolio of risky assets as wealth changes

Now -- many risky assets

- Theorem 4.6 (Cass and Stiglitz,1970). Let the vector

$$\begin{bmatrix} \hat{a}_1(Y_0) \\ . \\ . \\ \hat{a}_J(Y_0) \end{bmatrix}$$

denote the amount optimally invested in the *J* risky assets if the wealth level is $Y_{0.}$. Then

$$\begin{bmatrix} \hat{a}_1(Y_0) \\ . \\ . \\ \hat{a}_J(Y_0) \end{bmatrix} = \begin{bmatrix} a_1 \\ . \\ . \\ a_J \end{bmatrix} f(Y_0)$$

if and only if either

(i) $U'(Y_0) = (\theta Y_0 + \kappa)^{\Delta}$ or

(ii) $U'(Y_0) = \xi e^{-\nu Y_0}$.

- In words, it is sufficient to offer a **mutual fund.**

LRT/HARA-utility functions

- Linear Risk Tolerance/hyperbolic absolute risk aversion

$$-\frac{u''(c)}{u'(c)} = \frac{1}{A+Bc}$$

- Special Cases
  - B=0, A>0 CARA
  - B ≠ 0, ≠1 Generalized Power $u(c) = \frac{1}{B-1}(A+Bc)^{\frac{B-1}{B}}$
    - B=1 Log utility u(c) =ln (A+Bc)
    - B=-1 Quadratic Utility $u(c)=-(A-c)^2$
    - B ≠ 1 A=0 CRRA Utility function $u(c) = \frac{1}{B-1}(Bc)^{\frac{B-1}{B}}$

Prudence and Pre-cautionary Savings

- Introduce savings decision
  Consumption at $t=0$ and $t=1$
- Asset structure
  - NO risk free bond
  - One risky asset with random gross return R

Prudence and Savings Behavior

- Risk aversion is about the willingness to insure …
- … but not about its comparative statics.
- How does the behavior of an agent change when we marginally increase his exposure to risk?
- An old hypothesis (going back at least to J.M.Keynes) is that people should save more now when they face greater uncertainty in the future.
- The idea is called *precautionary saving* and has intuitive appeal.

Prudence and Pre-cautionary Savings

- Does not directly follow from risk aversion alone.
- Involves the third derivative of the utility function.
- Kimball (1990) defines **absolute prudence** as

$$P(w) := -u'''(w)/u''(w).$$

- Precautionary saving if any only if they are prudent.
- This finding is important when one does comparative statics of interest rates.
- Prudence seems uncontroversial, because it is weaker than DARA.

Pre-cautionary Saving

$$\max_s E[U(Y_0 - s) + \delta U(sR)]$$

$$\text{s.t. } s \geq 0$$

$$\text{FOC: } U'(Y_0 - \underset{(+)}{s}) = \delta E[U'(sR)R] \quad (-) \text{ in s}$$

Is saving *s* increasing/decreasing in risk of *R*?
Is RHS increasing/decreasing is riskiness of *R*?
Is *U'*() convex/concave?
Depends on third derivative of *U*()!

N.B: For *U*(*c*)=ln *c*, *U'*(*sR*)*R*=1/*s* does not depend on R.

Pre-cautionary Saving

2 effects: Tomorrow consumption is more volatile

- consume more today, since it's not risky
- save more for precautionary reasons

Theorem 4.7 (Rothschild and Stiglitz,1971) : Let $\tilde{R}_A$, $\tilde{R}_B$ be two return distributions with identical means such that $\tilde{R}_B = \tilde{R}_A + e$, (where *e* is white noise) and let s_A and s_B be, respectively, the savings out of Y_0 corresponding to the return distributions $\tilde{R}_A$ and $\tilde{R}_B$.

If $R'_R(Y) \leq 0$ and $R_R(Y) > 1$, then $s_A < s_B$;

If $R'_R(Y) \geq 0$ and $R_R(Y) < 1$, then $s_A > s_B$

Prudence & Pre-cautionary Saving

$$\mathbf{P}(c) = \frac{-U'''(c)}{U''(c)}$$

$$\mathbf{P}(c)c = \frac{-cU'''(c)}{U''(c)}$$

- Theorem 4.8 : Let $\tilde{R}_A, \tilde{R}_B$ be two return distributions such that $\tilde{R}_A$ SSD $\tilde{R}_B$, and let s_A and s_B be, respectively, the savings out of Y_0 corresponding to the return distributions $\tilde{R}_A$ and $\tilde{R}_B$. Then,

$s_A \geq s_B$ iff $c\mathbf{P}(c) \leq 2$, and conversely,

$s_A < s_B$ iff $c\mathbf{P}(c) > 2$

Joint saving-portfolio problem

- Consumption at *t=0* and *t=1*. (savings decision)
- Asset structure
  - – One risk free bond with net return r_f
  - – One risky asset (a = quantity of risky assets)

$$\max_{\{a,s\}} U(Y_0 - s) + \delta E U(s(1 + r_f) + a(\tilde{r} - r_f)) \qquad (4.7)$$

FOC:

s: $U'(c_t) = \delta\, E[U'(c_{t+1})(1+r_f)]$

a: $E[U'(c_{t+1})(r-r_f)] = 0$

for CRRA utility functions

$$s: \quad (Y_0 - s)^{-\gamma}(-1) + \delta E\left([s(1+r_f) + a(\tilde{r} - r_f)]^{-\gamma}(1+r_f)\right) = 0$$

$$a: \quad E\left[(s(1+r_f) + a(\tilde{r} - r_f))^{-\gamma}(\tilde{r} - r_f)\right] = 0$$

Where *s* is total saving and *a* is amount invested in risky asset.

Multi-period Setting

- Canonical framework (exponential discounting)

$$U(c) = E[\sum \beta^t u(c_t)]$$

  - prefers earlier uncertainty resolution if it affect action
  - indifferent, if it does not affect action
- Time-inconsistent (hyperbolic discounting)

  Special case: β–δ formulation

$$U(c) = E[u(c_0) + \beta \sum \delta^t u(c_t)]$$

- Preference for the timing of uncertainty resolution

  recursive utility formulation (Kreps-Porteus 1978)

Multi-period Portfolio Choice

$$\max_{\{s_t,a_t\}_{t=0}^{T-1}} E[\textstyle\sum_{t=0}^{T} \beta^t U(c_t)]$$

s.t.

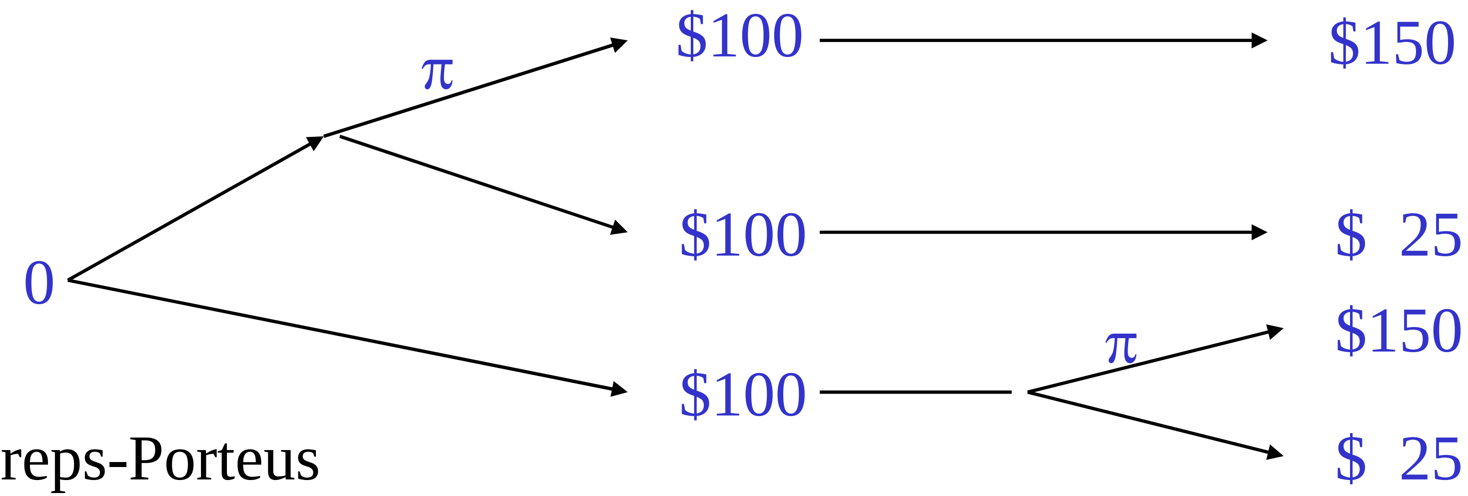
$$c_T = s_{T-1}(1+r_f) + a_{T-1}(r_T - r_f)$$

$$c_t + s_t \leq s_{t-1}(1+r_f) + a_{t-1}(r_t - r_f)$$

$$c_0 + s_0 \leq Y_0$$

Theorem 4.10 (Merton, 1971): Consider the above canonical multi-period consumption-saving-portfolio allocation problem. Suppose U() displays CRRA, r_f is constant and {r} is i.i.d. Then a/s_t is time invariant.

Digression: Preference for the timing of uncertainty resolution



Kreps-Porteus

$$U_0(x_1, x_2(s)) = W(x_1, E[U_1(x_1, x_2(s))])$$

Early (late) resolution if W(P_1,…) is convex (concave)

Marginal rate of temporal substitution 🕒 risk aversion