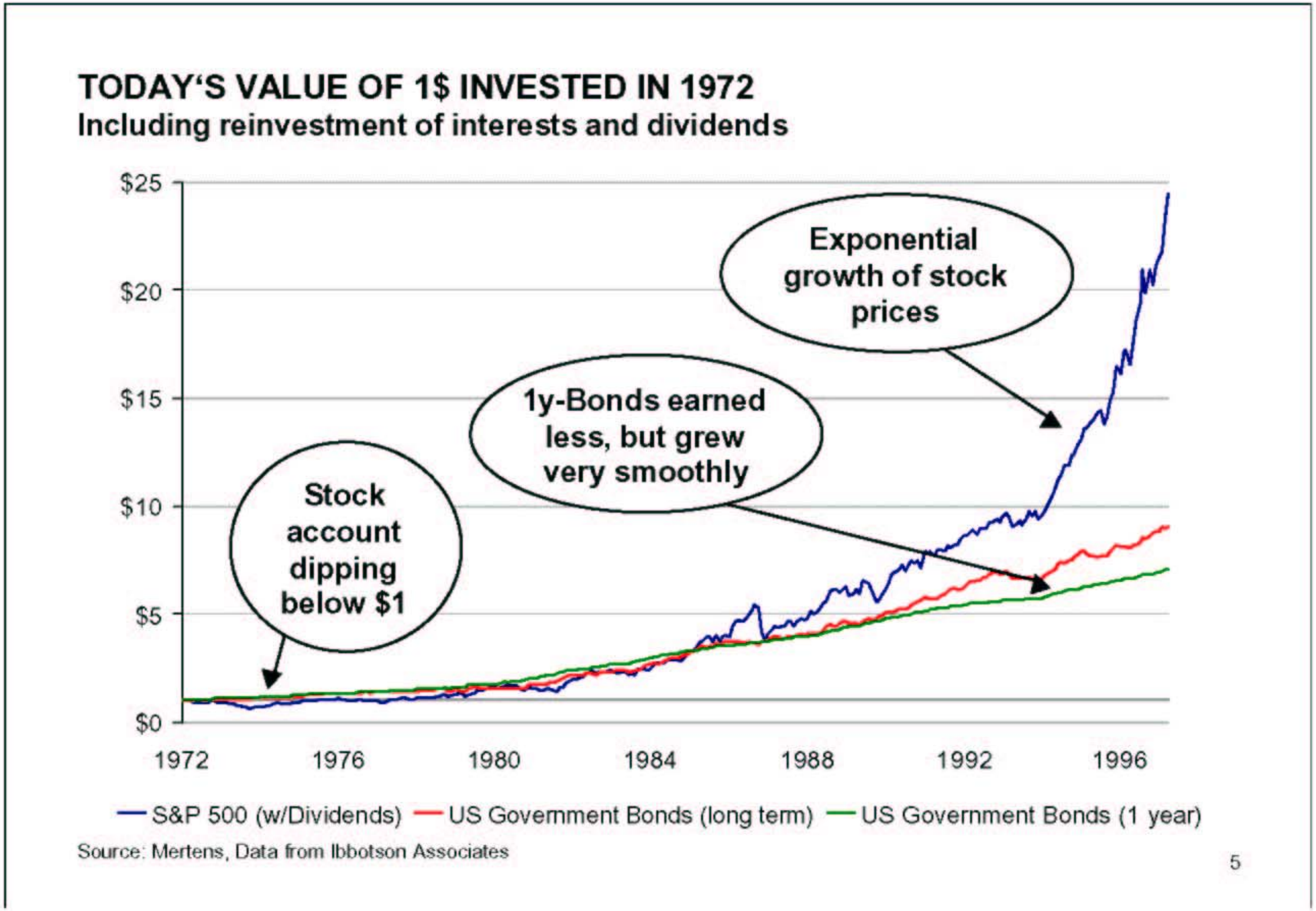
Lecture 03: Sharpe Ratio, Bounds and the Equity Premium Puzzle

Prof. Markus K. Brunnermeier

The long-term gains from the stock market have been astounding



Sharpe Ratios and Bounds

- Consider a one period security available at date t with payoff x_{t+1}. We have

  $$p_t = E_t[m_{t+1}\ x_{t+1}]$$

  or

  $$p_t = E_t[m_{t+1}]\ E_t[x_{t+1}] + Cov[m_{t+1},x_{t+1}]$$

- For a given m_{t+1} we let

  $$R^f_{t+1} = 1/\ E_t[m_{t+1}]$$

  – Note that R^f_t will depend on the choice of m_{t+1} unless there exists a riskless portfolio

Sharpe Ratios and Bounds (ctd.)

– Hence

$$p_t = (1/R^f_{t+1})\, E_t[x_{t+1}] + Cov[m_{t+1}, x_{t+1}]$$

– price = expected PV + Risk adjustment

– positive correlation with the discount factor adds value

in Returns

$$E_t\,[m_{t+1}\; x_{t+1}\;] = p_t$$

– divide both sides by p_t and note that $x_{t+1} = R_{t+1}$

$$E_t\,[m_{t+1}\; R_{t+1}] = 1 \qquad\qquad \text{(vector)}$$

– using $R^f_{t+1} = 1/\,E_t[m_{t+1}]$, we get

$$E_t\,[m_{t+1}\,(R_{t+1} - R^f_{t+1}\,)] = 0$$

– *m*-discounted expected excess return for all assets is zero.

in Returns

– Since $E_t\,[m_{t+1}\,(R_{t+1} – R^f_{t+1}\,)] = 0$

$$Cov_t[m_{t+1},R_{t+1}-R_t^{\,f}] \;=\; E_t[m_{t+1}(R_{t+1} – R_{t+1}^{\,f})] – E_t[m_{t+1}]E_t[R_{t+1} – R_{t+1}^{\,f}]$$

$$= - E_t[m_{t+1}]\; E_t[R_{t+1} – R_{t+1}^{\,f}]$$

- That is, risk premium or expected excess return

$$E_t\;[R_{t+1}-R_t^{\,f}] = - Cov_t[m_{t+1},R_{t+1}] \,/\, E[m_{t+1}]$$

is determined by its covariance with the stochastic discount factor

Sharpe Ratio

- Multiply both sides with portfolio h

$E_t\ [(R_{t+1}-R_t^f)h] = -\ Cov_t[m_{t+1},R_{t+1}h]\ /\ E[m_{t+1}]$

- NB: All results also hold for unconditional expectations E[·]

$$E[(R_{t+1}-R_t^f)h] = -\frac{[\rho(m_{t+1},R_{t+1}h)]\sigma(R_{t+1}h)\sigma(m_{t+1})}{E[m_{t+1}]}$$

- Rewritten in terms of Sharpe Ratio = ...

$$\frac{E[(R_{t+1}-R_t^f)h]}{\sigma(R_{t+1}h)} = -\frac{\sigma(m_{t+1})}{E[m_{t+1}]}[\rho(m_{t+1},R_{t+1}h)]$$

Hansen-Jagannathan Bound

– Since ρ ∈ [-1,1] we have

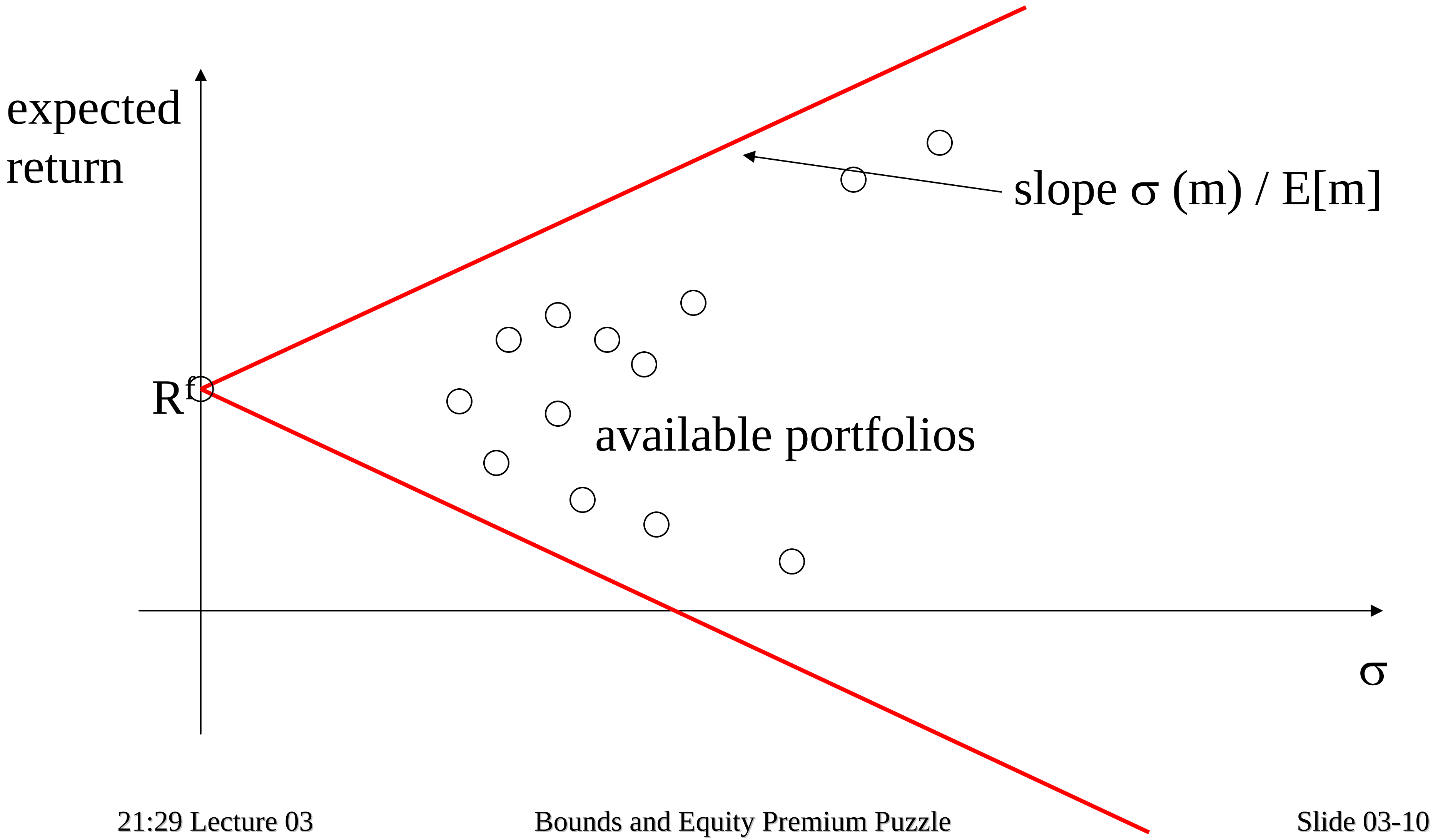
$$\frac{\sigma(m_{t+1})}{E[m_{t+1}]} \geq \sup_h \left| \frac{E[(R_{t+1} - R_t^f)h]}{\sigma(R_{t+1}h)} \right|$$

- **Theorem (Hansen-Jagannathan Bound):** The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.

Hansen-Jagannathan Bound

- **Theorem (Hansen-Jagannathan Bound):** The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
  - Can be used to easy check the “viability” of a proposed discount factor
  - Given a discount factor, this inequality bounds the available risk-return possibilities
  - The result also holds conditional on date *t* info

Hansen-Jagannathan Bound



Assuming Expected Utility

- $c_0 \in R$, $c_1 \in R^S$
- $U(c_0,c_1)= \sum_s \pi_s \, u(c_0,c_{1,s})$
- $$\partial_0 u = \left(\frac{\partial u(c_0^*,c_{1,1}^*)}{\partial c_0}, \ldots, \frac{\partial u(c_0^*,c_{1,S}^*)}{\partial c_0}\right)$$
  $$\partial_1 u = \left(\frac{\partial u(c_0^*,c_{1,1}^*)}{\partial c_{1,1}}, \ldots, \frac{\partial u(c_0^*,c_{1,S}^*)}{\partial c_{1,S}}\right)$$
- Stochastic discount factor

$$m = \left(\frac{MRS}{\pi}\right) = \left(\frac{\partial_1 u}{E[\partial_0 u]}\right) \in \mathbb{R}^S$$

- *Digression:* if utility is in addition time-separable

$$u(c_0,c_1) = v(c_0) + v(c_1)$$

- Then

$$\partial_0 u = \left(\frac{\partial v(c_0^*)}{\partial c_0}, \ldots, \frac{\partial v(c_0^*)}{\partial c_0}\right)$$

$$\partial_1 u = \left(\frac{\partial v(c_{1,1}^*)}{\partial c_{1,1}}, \ldots, \frac{\partial v(v_{1,S}^*)}{\partial c_{1,S}}\right)$$

- and

$$m_s = \frac{1}{\pi_s}\frac{\pi_s v'(c_{1,s})}{v'(c_0)} = \frac{v'(c_{1,s})}{v'(c_0)}$$

Equity Premium Puzzle

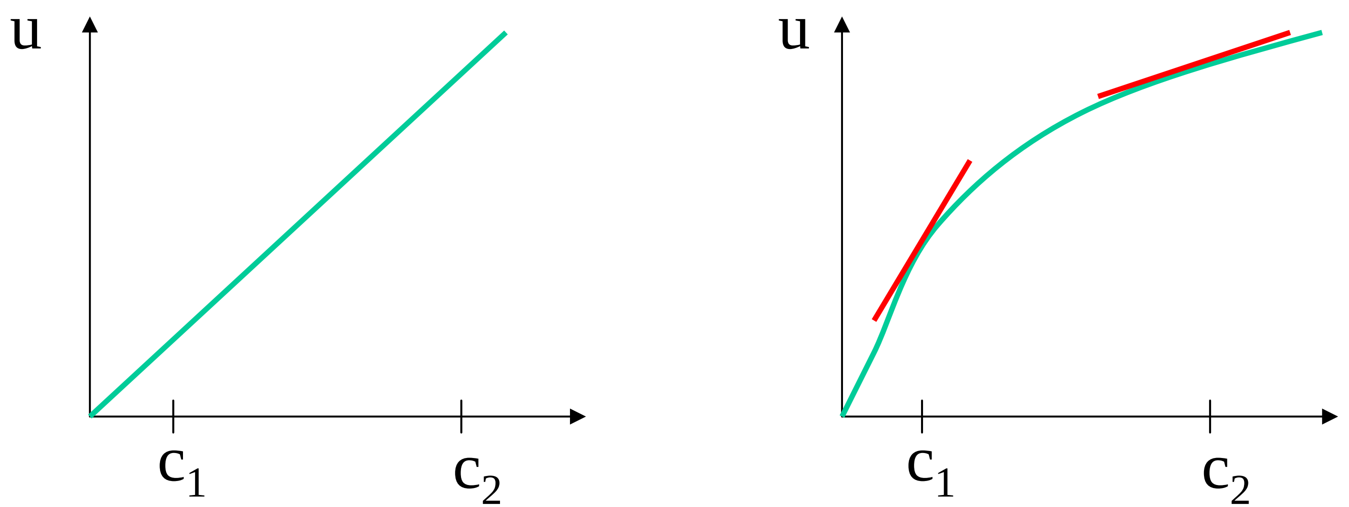
- Recall $E[R^j]-R^f = -R^f \, Cov[m,R^j]$
- Now: $E[R^j]-R^f = -R^f \, Cov[\partial_1 u,R^j]/E[\partial_0 u]$
- Recall Hansen-Jaganathan bound

$$\frac{\sigma(m)}{E[m]} \geq \left|\frac{E[(R-R^f)]}{\sigma(R)}\right|; \;\; E[m] = \frac{1}{R^f}$$

$$\sigma(m) \geq \frac{1}{R^f}\left|\frac{E[(R-R^f)]}{\sigma(R)}\right|$$

Equity Premium Puzzle (ctd.)

$$\sigma\left(\frac{\partial_1 u}{E[\partial_0 u]}\right) \geq \frac{1}{R^f}\left|\frac{E[(R-R^f)]}{\sigma(R)}\right|$$



Equity Premium Puzzle:

- high observed Sharpe ratio of stock market indices
- low volatility of consumption

$\Rightarrow$ (unrealistically) high level of risk aversion

A simple example

- $S=2$, $\pi_1 = ½$,
- 3 securities with $x^1 = (1,0)$, $x^2=(0,1)$, $x^3= (1,1)$
- Let $m=(½,1)$, $\sigma = ¼ =[½(½ - ¾)^2 + ½(1 - ¾)^2]^{½}$
- Hence, $p^1=¼$, $p^2= 1/2$, $p^3= ¾$ and
- $R^1 = (4,0)$, $R^2=(0,2)$, $R^3=(4/3,4/3)$
- $E[R^1]=2$, $E[R^2]=1$, $E[R^3]=4/3$

Example: Where does SDF come from?

- "representative agent" with
  - endowment: 1 in date 0, (2,1) in date 1
  - utility $EU(c_0, c_1, c_2) = \sum_s \pi_s (\ln c_0 + \ln c_{1,s})$
  - i.e. $u(c_0, c_{1,s}) = \ln c_0 + \ln c_{1,s}$ (additive) time separable u-function
- $m= \partial_1 u\ (1,2,1) / E[\partial_0\ u(1,2,1)]=(c_0/c_{1,1},\ c_0/c_{1,2})=(1/2,\ 1/1)$
- m=(½,1) since endowment=consumption
- Low consumption states are high "m-states"
- Risk-neutral probabilities combine true probabilities and marginal utilities.