# Asset Pricing under Asymmetric Information
# Modeling Information & Solution Concepts

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# References

**Books:**

Brunnermeier (2001), "Asset Pricing under Asym. Info."
Vives (2006), "Information and Learning in Markets"
O'Hara (1995), "Market Microstructure Theory"

**Artciles:**

many - see syllabus

Some parts of these slides rely on Princeton lecture notes by Nöldeke (1993)

# Two Interpretations of Asymmetric Information

- different information
- different interpretation of the same information (different background information)

# Modeling information I

- State space $\Omega$
  - state $\omega \in \Omega$ = full description of reality
    - fundamentals
    - signals
  - state space is common knowledge and fully agreed among agents

# Modeling information II

- Partition
  - $(\omega_1, \omega_2, \omega_3), (\omega_4, \omega_5), (\omega_6, \omega_7, \omega_8)$
  - $\mathcal{P}_1^i, \mathcal{P}_2^i, \mathcal{P}_3^i$ (partition cells)
  - later more about ‘knowledge operators’ etc.
- Field (Sigma-Algebra) $\mathcal{F}^i$
- Probability measure/distribution $P$

# Modeling information III

- Prior distribution
  - Common prior assumption (CPA) (Harsanyi doctrine)
    - any difference in beliefs is due to differences in info
    - has strong implications
  - Rational Expectations
    - $\text{prior}^i$ = objective distribution $\forall i$
    - implies CPA
  - Non-common priors
    - Problem: almost everything goes
    - Way out: Optimal Expectations
      (structure model of endogenous priors)
- Updating/Signal Extraction

# Modeling information III

- Updating (general)
  - Bayes' Rule

$$P^i\left(E_n|D\right)=\frac{P^i\left(D|E_n\right)P^i\left(E_n\right)}{P^i\left(D\right)},$$

  - if events $E_1, E_2, ..., E_N$ are a partition

$$P^i\left(E_n|D\right)=\frac{P^i\left(D|E_n\right)P^i\left(E_n\right)}{\sum_{n=1}^{N}P^i\left(D|E_n\right)P^i\left(E_n\right)},$$

# Updating - Signal Extraction - general case

- Updating - Signal Extraction
  - $\omega = \{v, S\}$
  - desired property:
    signal realization $S^H$ is always more favorable than $S^L$
  - formally: $G(v|S^H)$ FOSD $G(v|S^L)$
  - Milgrom (1981) shows that this is equivalent to $f_S(S|v)$ satisfies monotone likelihood ratio property (MLRP)
  - $f_S(S|v)/f_S(S|\bar{v})$ is increasing (decreasing) in $S$ if $v > (<) \bar{v}$

$$\frac{f_S(S|v)}{f_S(S|v')} > \frac{f_S(S'|v)}{f_S(S'|v')} \forall v' > v \text{ and } S' > S.$$

  - another property:
    hazard rate $\frac{f_S(S|v)}{1-F(S|v)}$ is declining in $v$

# Updating - Signal Extraction - Normal distributions

- updating normal variable $X$ after receiving signal $S = s$

$$\begin{aligned} E[X|S=s] &= E[X] + \frac{Cov[X,S]}{Var[S]}(s - E[S]) \\ Var[X|S=s] &= Var[X] - \frac{Cov[X,S]^2}{Var[S]} \end{aligned}$$

- $n$ multidimensional random variable $\left(\vec{X}, \vec{S}\right) \sim \mathcal{N}(\mu, \Sigma)$

$$\mu = \begin{bmatrix} \mu_X \\ \mu_S \end{bmatrix}_{n\times 1}; \Sigma = \begin{bmatrix} \Sigma_{X,X} & \Sigma_{X,S} \\ \Sigma_{S,X} & \Sigma_{S,S} \end{bmatrix}_{n\times n}.$$

- Projection Theorem $(X|S=s)$

$$\sim \mathcal{N}\left(\mu_X + \Sigma_{X,S}\Sigma_{S,S}^{-1}(s - \mu_S), \Sigma_{X,X} - \Sigma_{X,S}\Sigma_{S,S}^{-1}\Sigma_{S,X}\right)$$

# Special Signal Structures

- $\mathcal{N}$-Signals of form: $S_n = X + \varepsilon_n$
  (Let X be a scalar and $\tau_y = \frac{1}{Var[y]}$),

$$E[X|s_1, ..., s_N] = \mu_X + \frac{1}{\tau_X + \sum_{n=1}^N \tau_{\varepsilon_n}} \sum_{n=1}^N \tau_{\varepsilon_n} (s_n - \mu_X)$$

$$Var[X|s_1, ..., s_N] = \frac{1}{\tau_X + \sum_{n=1}^N \tau_{\varepsilon_n}} = \frac{1}{\tau_{X|s_1,...,s_N}}$$

  - If, in addition, all $\varepsilon_n$ i.i.d. then

$$E[X|s_1, ..., s_N] = \mu_X + \underbrace{\frac{1}{\tau_X + N\tau_{\varepsilon_n}}}_{Var[X|s_1,...,s_N]} N\tau_{\varepsilon_n} \left( \sum_{n=1}^N \frac{1}{N} s_n - \mu_X \right),$$

where $\bar{s} := \sum_{n=1}^N \left(\frac{1}{N}\right) s_n$ is a *sufficient statistic*

# Special Signal Structures

- $\mathcal{N}$-Signals of form: $X = S + \varepsilon$

$$\begin{aligned} E[X|S=s] &= s \\ Var[X|S=s] &= Var[\varepsilon] \end{aligned}$$

- Binary Signal: Updating with binary state space/signal
  - $q$ = precision = prob$(X = H | S = S^H)$
- "Truncating signals": $v \in [\overline{S}, \underline{S}]$
  - $v$ is Laplace (double exponentially) distributed or uniform
  - posterior is a truncated exponential or uniform

  (see e.g. Abreu & Brunnermeier 2002, 2003)

# Solution/Equilibrium Concepts

- Rational Expectations Equilibrium
    - Competitive environment
    - agents take prices as given (price takers)
    - Rational Expectations (RE) $\Rightarrow$ CPA
    - *General Equilibrium Theory*
- Bayesian Nash Equilibrium
    - Strategic environment
    - agents take strategies of others as given
    - CPA (RE) is typically assumed
    - *Game Theory*
    - distinction between normal and extensive form games simultaneous move versus sequential move

# The 5 Step Approach

| | REE | BNE (sim. moves) |
|---|---|---|
| **Step 1** | Specify joint priors<br>Conject. price mapping**s**<br>$P : \{\mathcal{S}^1, ..., \mathcal{S}^I, u\} \to \mathbb{R}_+^{\mathbb{J}}$ | Specify joint priors<br>Conjecture strategy profile**s** |
| **Step 2** | Derive posteriors | Derive posteriors |
| **Step 3** | Derive individual demand | Derive best response |
| **Step 4** | Impose market clearing | |
| **Step 5** | Impose Rationality<br>Equate undet. coeff. | Impose Rationality<br>No-one deviates |

# A little more abstract

- REE
  Fixed Point of Mapping: $\mathcal{M}_P(P(\cdot)) \mapsto P(\cdot)$
- BNE (simultaneous moves)
  Fixed Point of Mapping:
  strategy profiles $\mapsto$ strategy profiles
- What's different for sequential move games?
  - late movers react to deviation
  - equilibrium might rely on 'strange' out of equilibrium moves
  - refinement: subgame perfection
- Extensive form move games with asymmetric information
  - Sequential equilibrium (agents act sequentially rational)
  - Perfect BNE (for certain games)
    - nature makes a move in the beginning (chooses type)
    - action of agents are observable

# A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models:
    (uninformed) market maker submits a supply schedule first
    - static
      - ◇ uniform price setting
      - ◇ limit order book analysis
    - dynamic sequential trade models with multiple trading rounds
  - signalling models:
    informed traders move first, market maker second