Problem Set 2

Fin 525: Financial Economics I

Part 1: Asset Pricing in Discrete Time

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Due Date: Monday, October 2

Nota Bene: Please do not feel obliged to solve all of these problems...this problem set, like most, is long and only for your benefit. Do what interests and engages you, as well as helps you learn the material. However, we strongly recommend you try your hand at problems 4 and 5 as they are a bit harder and teach you something.

Problem 1

Let Ω be the (finite) set of states of the world tomorrow and suppose that an individual chooses among actions $a \in A$ so as to maximize $E[u(a,\omega)]$, where the expectation is taken with respect to the distribution $P(\cdot)$ over possible future states of the world. Now suppose another individual has utility function $v(a,\omega) = \alpha u(a,\omega) + \beta, \forall a \in A, \omega \in \Omega$ with $\alpha > 0$.

a) Show that if the second individual also maximizes her expected utility, then she will make the exact same choice as the first individual; in fact, show that any two actions $a, a' \in A$ will be ranked (in terms of their expected utility) identically by the two individuals. What does this tell us about utility functions? Up to what sort of transformations are they defined in terms of their positive implications? Might parameters of a utility function that have no *positive* implications matter in some types of analysis?

b) Provide an example where v is a *positive monotonic transformation* of u in the sense that $v(a,\omega) > v(a',\omega') \iff u(a,\omega) > u(a',\omega')$ but nonetheless individual two makes different choices than individual one. Why can expected utility vary under this sort of transformation? Would a change like this make any difference under certainty?

c) In the context of finance, what sort of things do with A and Ω represent?

Problem 2

Consider the following utility functions (defined over wealth Y):

$$
\begin{aligned}
(1)\ U(Y) &= -\frac{1}{Y} \\
(2)\ U(Y) &= \ln(Y) \\
(3)\ U(Y) &= -Y^{-\gamma} \\
(4)\ U(Y) &= -\exp(-\gamma Y) \\
(5)\ U(Y) &= \frac{Y^{\gamma}}{\gamma} \\
(6)\ U(Y) &= \alpha Y - \beta Y^2
\end{aligned}
$$

*Grader: E. Glen Weyl

a) Check that they are well behaved ($U' > 0, U'' < 0$) or state restrictions on the parameters so that they are (utility functions (1) – (5)). For utility function (6), take positive α and β, and give the range of wealth over which the utility function is well behaved.

b) Compute the absolute and relative risk aversion coefficients. Are all of these utility functions hyperbolic average risk aversion (HARA)? For those that are, calculate the linear equation for their risk tolerance as a function of wealth.

c) What is the effect of the parameter γ (when relevant)?

d) Classify the functions as increasing/decreasing risk aversion utility functions (both absolute and relative). How does this relate to the risk tolerance specification discussed above?

Problem 3

Consider an economy with two types of financial assets: one risk-free and one risky asset. The rate of return offered by the risk-free asset is r_f. The rate of return of the risky asset is $\tilde{r}$. Note that the expected rate of return $E(\tilde{r}) > r_f$.

Agents are risk-averse. Let Y_0 be the initial wealth. The purpose of this exercise is to determine the optimal amount a to be invested in the risky asset as a function of the Arrow-Pratt measure of absolute risk aversion.

The objective of the agents is to maximize the expected utility of terminal wealth:

$$\max_a E(U(Y))$$

where E is the expectation operator, $U(\cdot)$ is the utility function with $U' > 0$ and $U'' < 0$, Y is the wealth at the end of the period, and a is the amount being invested in the risky asset.

a) Determine the final wealth as a function of a, r_f, and $\tilde{r}$.

b) Compute the f.o.c. (first order condition). Is this a maximum or a minimum?

c) We are interested in determining the sign of da^*/dY_0. Calculate first the total differential of the f.o.c. as a function of a and Y_0. Write the expression for da^*/dY_0. Show that the sign of this expression depends on the sign of its numerator.

d) You know that R_A, the absolute risk aversion coefficient, is equal to $-U''(\cdot)/U'(\cdot)$. What does it mean if $R'_A = dR_A/dY < 0$?

e) Assuming $R'_A < 0$, compare $R_A(Y)$ and $R_A(Y_0(1+r_f))$: Is $R_A(Y) > R_A(Y_0(1+r_f))$ or vice-versa? Don't forget there are two possible cases: $\tilde{r} \geq r_f$ and $\tilde{r} < r_f$.

f) Show that

$$U''(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f) \geq -R_A(Y_0(1+r_f)) \times U'(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)$$

for both cases in part e).

g) Finally, compute the expectation of $U''(Y)(\tilde{r} - r_f)$. Using the f.o.c., determine its sign. What can you conclude about the sign of da^*/dY_0? What was the key assumption for the demonstration?

Problem 4

This problem is inspired by Rabin (2000). Expected utility has long been criticized for being unable to capture individuals' risk preferences over gambles that are small relative to their wealth. In this problem, we will consider whether this critique.

a) Suppose that I offer you a choice between \$.50 cents and \$1.01 with %50 probability. Which would you choose? If you did not choose the \$.50, do you think this is a reasonable choice? Would it be reasonable to make this choice, regardless of your level of wealth?

b) Suppose, now, that in fact you would choose the \$.50 at any level of wealth; even if this is not your preference, it is expressed by most experimental subjects. Suppose, too, that your choices to maximize the expected utility of wealth, with a concave (risk-averse) utility function. Approximately how much lower must your marginal utility of wealth be at a wealth \$1 above your current wealth than at your current wealth for you to make this choice? That is, if you wealth is w, what is an upper bound on $\frac{u'(w+1)}{u'(w)}$?

c) What is an upper bound on $\frac{u'(10000)}{u'(5000)}$? Now suppose you have \$1000 of wealth and you make choices according to these preferences. Would you prefer to receive \$9000 with certainty or \$100,000,000,000 with probability .99999999? How can you determine this (hint: figure out what know about lower and upper bounds on marginal utility of wealth at different levels of wealth)?

d) Does this preference seem plausible? Do you think that expected utility is a good positive model of risk preferences over small gambles? What challenges does this pose for the models of financial choice we consider?

e) On the other hand, do you think expected utility is a good *normative* model of decision making under risk, and particularly investment? Has learning about expected utility (in this problem, this course and elsewhere) shifted how you think about risk? Suppose that, as a financial economist, someone came to you for detailed advice about investing. Presumably you would try to get some sense of their risk preferences; do you think it would be reasonable, though, to assume they are (want to be) expected utility maximizers? Should you "impose" this normative assumption on them? If so, what do you find compelling about expected utility? If not, what assumptions can you make about their preferences in trying to determine how you should advise them? This is a philosophical question and there is no right answer, but it is important for economists to think about this sort of question.

Problem 5

There are two individuals (1 and 2). $u_i(w_i) = \frac{w_i^{1-\gamma_1}}{1-\gamma_1}$ for both of them, but $\gamma_1 < \gamma_2$. There is no consumption today and there are two states of the world tomorrow, each occurring with probability .5: good and bad. In the good state, each individual is endowed with two units of consumption. In the bad state, each individual is endowed with one unit. Suppose that markets are complete.

a) First consider the two possible "thievery" equilibria. Suppose that individual 1 is able to steal all of individual 2's endowment and become the "representative agent". What state prices prevail (cause individual 1 to be indifferent between buying or selling consumption in each state to herself)? Suppose, instead, that individual 2 takes all of 1's endowment, now what state prices prevail? (Hint: you should only need one calculation for both cases).

b) Now consider the actual equilibrium in the market where the two individuals trade. Which individual do you think will take on more risk (from their utility function) and why? Derive the equilibrium. Were you right? The individuals had exactly the same endowment. Which had a greater impact on the equilibrium prices? Is this intuitive? Does this suggest a general relationship between risk preference and the impact investors have on prices? Provide some economic intuition.