Lecture 3: One-period Model Pricing

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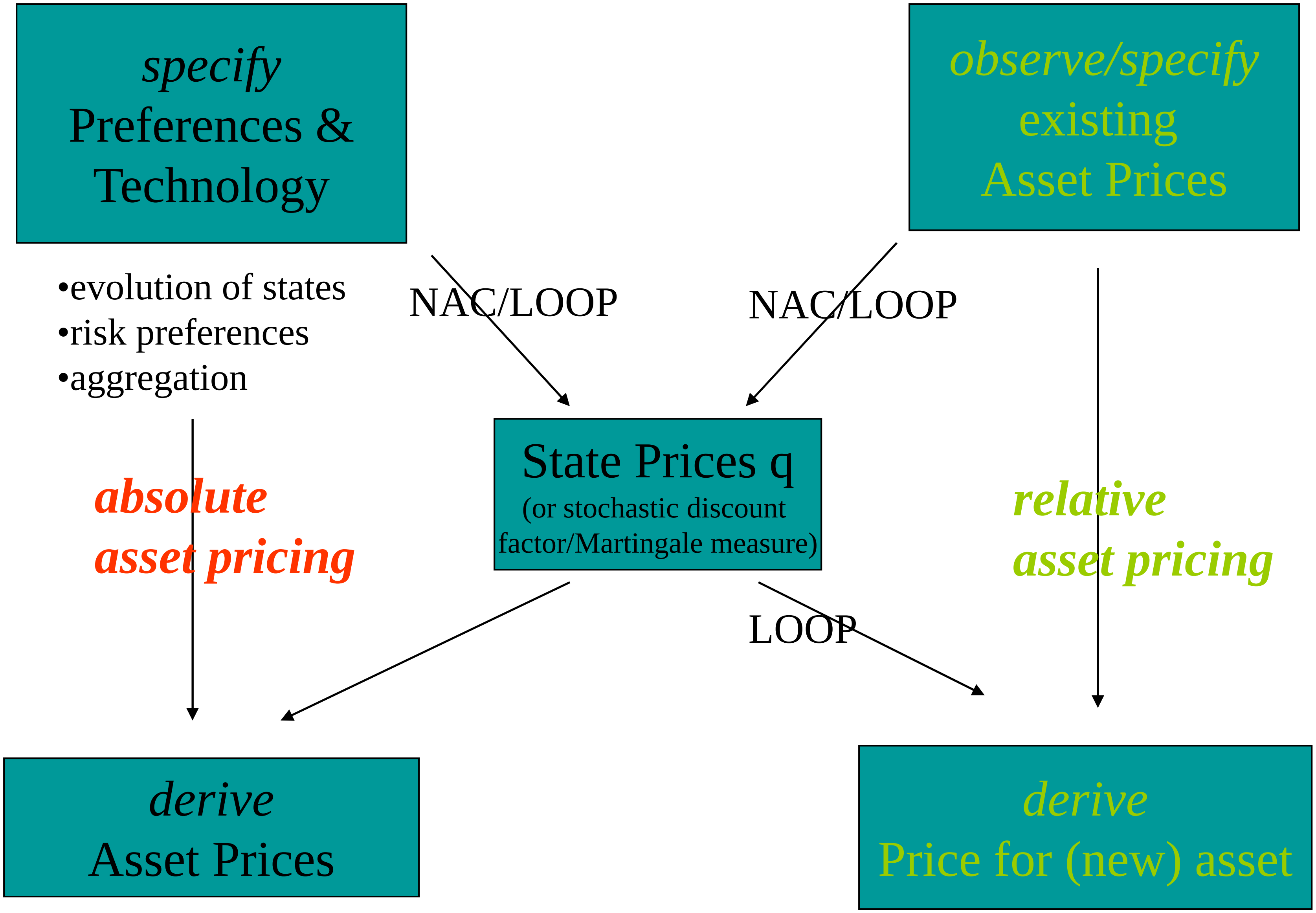
Overview: Pricing

1. LOOP, No arbitrage [L2,3]
2. Forwards [McD5]
3. Options: Parity relationship [McD6]
4. No arbitrage and existence of state prices [L2,3,5]
5. Market completeness and uniqueness of state prices
6. Pricing kernel q^*
7. Four pricing formulas:
   state prices, SDF, EMM, beta pricing [L2,3,5,6]
8. Recovering state prices from options [DD10.6]

Vector Notation

- Notation: $y,x \in R^n$
  - $y \geq x \Leftrightarrow y^i \geq x^i$ for each i=1,…,n.
  - $y > x \Leftrightarrow y \geq x$ and $y \neq x$.
  - $y >> x \Leftrightarrow y^i > x^i$ for each i=1,…,n.
- Inner product
  - $y \leq x = \sum_i yx$
- Matrix multiplication



Only works as long as market completeness doesn't change

Three Forms of No-ARBITRAGE

1. Law of one Price (LOOP)
   If $h'X = k'X$ then $p \leq h = p \leq k$.
2. No strong arbitrage
   There exists no portfolio h which is a strong arbitrage, that is $h'X \geq 0$ and $p \leq h < 0$.
3. No arbitrage
   There exists no strong arbitrage
   nor portfolio k with $k' X > 0$ and $0 \geq p \leq k$

Three Forms of No-ARBITR AGE

- Law of one price is equivalent to
  every portfolio with zero payoff has zero price.
- No arbitrage => no strong arbitrage
  No strong arbitrage => law of one price

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Alternative ways to buy a stock

- Four different payment and receipt timing combinations:
  - Outright purchase: ordinary transaction
  - Fully leveraged purchase: investor borrows the full amount
  - Prepaid forward contract: pay today, receive the share later
  - Forward contract: agree on price now, pay/receive later
- Payments, receipts, and their timing:

TABLE 5.1 Four different ways to buy a share of stock that has price S_0 at time 0. At time 0 you agree to a price, which is paid either today or at time T. The shares are received either at 0 or T. The interest rate is r.

Description	Pay at Time:	Receive Security at Time:	Payment
Outright Purchase	0	0	S_0 at time 0
Fully Leveraged Purchase	T	0	$S_0 e^{rT}$ at time T
Prepaid Forward Contract	0	T	?
Forward Contract	T	T	$? \times e^{rT}$

Pricing prepaid forwards

- If we can price the *prepaid* forward (F^P), then we can calculate the price for a forward contract:

$$F = \text{Future value of } F^P$$

- Pricing by analogy
  - In the absence of dividends, the timing of delivery is irrelevant
  - Price of the prepaid forward contract same as current stock price
  - $F^P_{0,T} = S_0$ (where the asset is bought at $t = 0$, delivered at $t = T$)

Pricing prepaid forwards (cont.)

- Pricing by arbitrage
  - If at time t=0, the prepaid forward price somehow exceeded the stock price, i.e., $F^P_{0,T} > S_0$, an arbitrageur could do the following:

TABLE 5.2 Cash flows and transactions to undertake arbitrage when the prepaid forward price, $F^P_{0,T}$, exceeds the stock price, S_0.

	Cash Flows	
Transaction	**Time 0**	**Time T (expiration)**
Buy Stock @ S_0	$-S_0$	$+S_T$
Sell Prepaid Forward @ $F^P_{0,T}$	$+F^P_{0,T}$	$-S_T$
Total	$F^P_{0,T} - S_0$	0

Pricing prepaid forwards (cont.)

- What if there are deterministic* dividends? Is $F^P_{0,T} = S_0$ still valid?
  - No, because the holder of the forward will not receive dividends that will be paid to the holder of the stock → $F^P_{0,T} < S_0$

$$F^P_{0,T} = S_0 - \text{PV}(\text{all dividends paid from t=0 to t=T})$$

  - For discrete dividends D_{t_i} at times t_i, $i = 1,\ldots, n$
    - The prepaid forward price: $F^P_{0,T} = S_0 - \sum^n_{i=1} PV_{0,t_i}(D_{t_i})$ (reinvest the dividend at risk-free rate)
  - For continuous dividends with an annualized yield δ
    - The prepaid forward price: $F^P_{0,T} = S_0 e^{-\delta T}$ (reinvest the dividend in this index. One has to invest only $S_0 e^{-\delta T}$ initially)

* NB 1: if dividends are stochastic, we cannot apply the one period model

Pricing prepaid forwards (cont.)

- Example 5.1
  - XYZ stock costs \$100 today and will pay a quarterly dividend of \$1.25. If the risk-free rate is 10% compounded continuously, how much does a 1-year prepaid forward cost?
  - $F^P_{0,1} = \$100 - \sum_{i=1}^{4} \$1.25e^{-0.025i} = \$95.30$
- Example 5.2
  - The index is \$125 and the dividend yield is 3% continuously compounded. How much does a 1-year prepaid forward cost?
  - $F^P_{0,1} = \$125e^{-0.03} = \$121.31$

Pricing forwards on stock

- Forward price is the future value of the *prepaid* forward
  - No dividends: $F_{0,T} = FV(F^P_{0,T}) = FV(S_0) = S_0\, e^{rT}$
  - Discrete dividends: $F_{0,T} = S_0\, e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)} D_{t_i}$
  - Continuous dividends: $F_{0,T} = S_0\, e^{(r-\delta)T}$
- Forward premium
  - The difference between current forward price and stock price
  - Can be used to infer the current stock price from forward price
  - Definition:
    - Forward premium = $F_{0,T} / S_0$
    - Annualized forward premium =: $\pi^a = (1/T)\, ln\, (F_{0,T} / S_0)$ (from $e^{\pi T} = F_{0,T} / S_0$)

Creating a *synthetic* forward

- One can offset the risk of a forward by creating a *synthetic* forward to offset a position in the actual forward contract
- How can one do this? (assume continuous dividends at rate δ)
  - Recall the long forward payoff at expiration: = $S_T - F_{0,T}$
  - Borrow and purchase shares as follows:

TABLE 5.3 Demonstration that borrowing $S_0 e^{-\delta T}$ to buy $e^{-\delta T}$ shares of the index replicates the payoff to a forward contract, $S_T - F_{0,T}$.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Buy $e^{-\delta T}$ Units of the Index	$-S_0 e^{-\delta T}$	$+S_T$
Borrow $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$-S_0 e^{(r-\delta)T}$
Total	0	$S_T - S_0 e^{(r-\delta)T}$

  - Note that the total payoff at expiration is same as forward payoff

Creating a *synthetic* forward (cont.)

- The idea of creating synthetic forward leads to following:
  - Forward = Stock – zero-coupon bond
  - Stock = Forward + zero-coupon bond
  - Zero-coupon bond = Stock – forward
- Cash-and-carry arbitrage: Buy the index, short the forward

TABLE 5.6 Transactions and cash flows for a cash-and-carry: A market-maker is short a forward contract and long a synthetic forward contract.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Buy Tailed Position in Stock, Paying $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short Forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

Other issues in forward pricing

- Does the forward price predict the future price?
  - According the formula $F_{0,T} = S_0 e^{(r-\delta)T}$ the forward price conveys no additional information beyond what S_0, r, and δ provides
  - Moreover, the forward price underestimates the future stock price
- Forward pricing formula and cost of carry
  - Forward price =

Spot price + Interest to carry the asset – asset lease rate

Cost of carry, $(r-\delta)S$ (brace under "Interest to carry the asset – asset lease rate")

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Put-Call Parity

- For European options with the same strike price and time to expiration the parity relationship is:

Call – put = *PV* (forward price – strike price)

or

$$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$$

- Intuition:
  - Buying a call and selling a put with the strike equal to the forward price ($F_{0,T} = K$) creates a synthetic forward contract and hence must have a zero price.

Parity for Options on Stocks

- If underlying asset is a stock and *Div* is the deterministic* dividend stream, then $e^{-rT} F_{0,T} = S_0 - PV_{0,T}(Div)$, therefore

$$C(K, T) = P(K, T) + [S_0 - PV_{0,T}(Div)] - e^{-rT}(K)$$

- Rewriting above,

$$S_0 = C(K, T) - P(K, T) + PV_{0,T}(Div) + e^{-rT}(K)$$

- For index options, $S_0 - PV_{0,T}(Div) = S_0 e^{-\delta T}$, therefore

$$C(K, T) = P(K, T) + S_0 e^{-\delta T} - PV_{0,T}(K)$$

*allows to stick with one period setting

Option price boundaries

- American vs. European
  - Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option:

$$C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T)$$

$$P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T)$$

- Option price boundaries
  - Call price cannot: be negative, exceed stock price, be less than price implied by put-call parity using zero for put price:

$$S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) >$$
$$> max\, [0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]$$

Option price bounderies (cont.)

- Option price boundaries
  - Call price cannot:
    - be negative
    - exceed stock price
    - be less than price implied by put-call parity using zero for put price:

    $S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq max\ [0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]$

  - Put price cannot:
    - be more than the strike price
    - be less than price implied by put-call parity using zero for call price:

    $K > P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq max\ [0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})]$

Early exercise of American call

- Early exercise of American options
  - A non-dividend paying American call option should not be exercised early, because:

$$C_{\text{Amer}} \geq C_{\text{Eur}} \geq S_t - K + P_{Eur} + K(1-e^{-r(T-t)}) > S_t - K$$

  - That means, one would lose money be exercising early instead of selling the option
  - If there are dividends, it may be optimal to exercise early
  - It may be optimal to exercise a non-dividend paying put option early if the underlying stock price is sufficiently low

Options: Time to expiration

- Time to expiration
  - An American option (both put and call) with more time to expiration is at least as valuable as an American option with less time to expiration. This is because the longer option can easily be converted into the shorter option by exercising it early.
  - European call options on dividend-paying stock and European puts may be less valuable than an otherwise identical option with less time to expiration.
  - A European call option on a non-dividend paying stock will be more valuable than an otherwise identical option with less time to expiration.
    - Strike price does not grow at the interest rate.
  - When the strike price grows at the rate of interest, European call and put prices on a non-dividend paying stock increases with time.
    - Suppose to the contrary P(T) < P(t) for T>t, then arbitrage. Buy P(T) and sell P(t) initially. At t
      - if $S_t > K_t$, P(t)=0.
      - if $S_t < K_t$, negative payoff $S_t - K_t$. Keep stock and finance K_t.
      - Time T-value $S_T - K_t e^{r(T-t)} = S_T - K_T$.

Options: Strike price

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - A call with a low strike price is at least as valuable as an otherwise identical call with higher strike price:

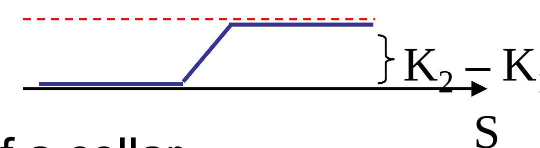
$$C(K_1) \geq C(K_2)$$

  - A put with a high strike price is at least as valuable as an otherwise identical call with low strike price:

$$P(K_2) \geq P(K_1)$$

  - The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices:

$$C(K_1) - C(K_2) \leq K_2 - K_1$$



Price of a collar ≤ maximum payoff of a collar

Note for K_3-K_2 more pronounced, hence (next slide) …!

Options: Strike price (cont.)

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - The premium difference between otherwise identical puts with different strike prices cannot be greater than the difference in strike prices:

$$P(K_1) - P(K_2) \leq K_2 - K_1$$

  - Premiums decline at a decreasing rate for calls with progressively higher strike prices. (Convexity of option price with respect to strike price):

$$\frac{C(K_1) - C(K_2)}{K_1 - K_2} \leq \frac{C(K_2) - C(K_3)}{K_2 - K_3}$$

Options: Strike price

TABLE 9.7 The example in Panel A violates the proposition that the rate of change of the option premium must decrease as the strike price rises. The rate of change from 50 to 59 is 5.1/9, while the rate of change from 59 to 65 is 3.9/6. We can arbitrage this convexity violation with an asymmetric butterfly spread. Panel B shows that we earn at least $3 plus interest at time T.

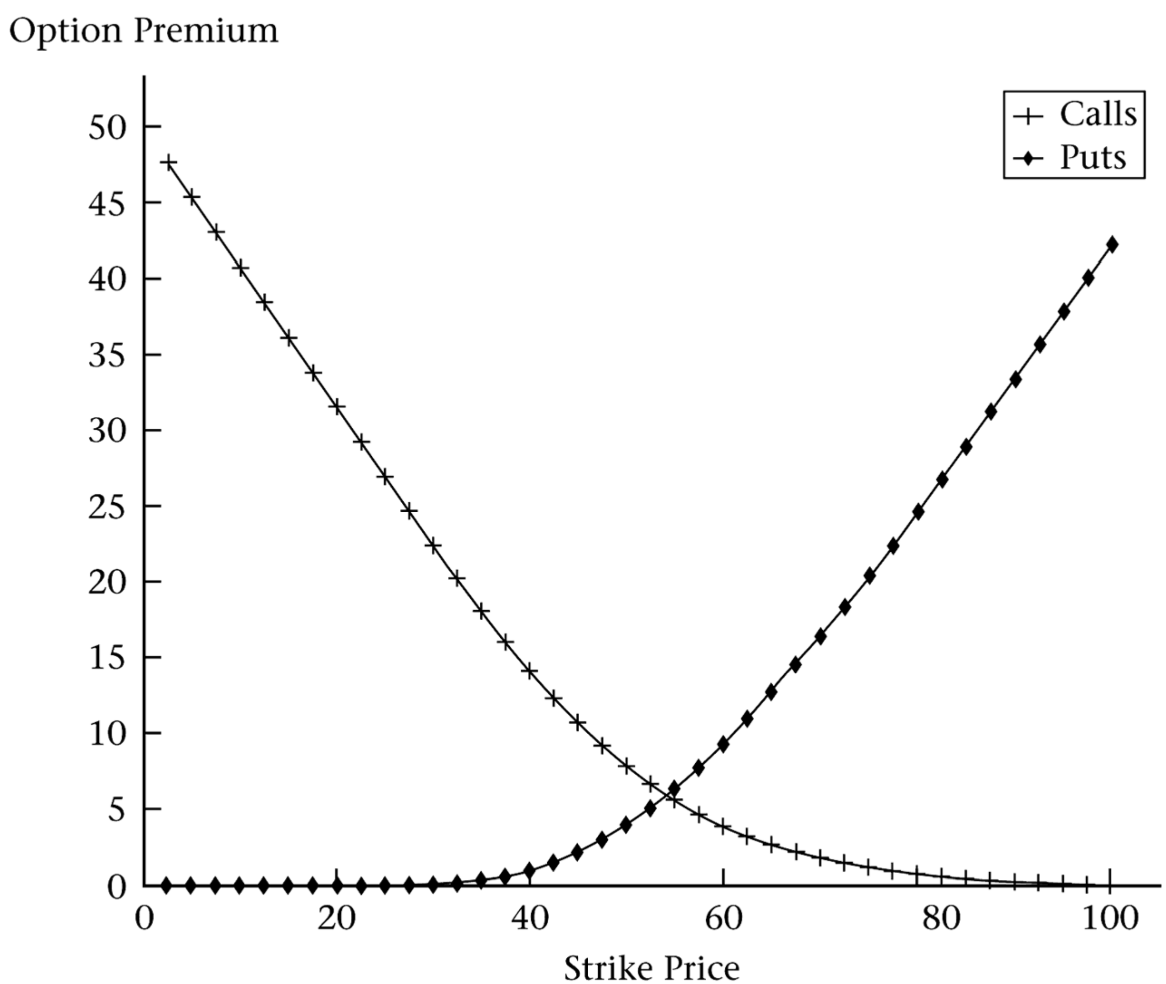
Panel A

Strike	50	59	65
Call Premium	14	8.9	5

Panel B

Transaction	Time 0	Expiration or Exercise			
		$S_T < 50$	$50 \le S_T \le 59$	$59 \le S_T \le 65$	$S_T > 65$
Buy Four 50-Strike Calls	-56	0	$4(S_T - 50)$	$4(S_T - 50)$	$4(S_T - 50)$
Sell Ten 59-Strike Calls	89	0	0	$10(59 - S_T)$	$10(59 - S_T)$
Buy Six 65-Strike Calls	-30	0	0	0	$6(S_T - 65)$
Lend $3	-3	$3e^{rT}$	$3e^{rT}$	$3e^{rT}$	$3e^{rT}$
Total	0	$3e^{rT}$	$3e^{rT} + 4(S_T - 50)$	$3e^{rT} + 6(65 - S_T)$	$3e^{rT}$

Properties of option prices (cont.)



Summary of parity relationships

TABLE 9.9 Versions of put-call parity. Notation in the table includes the spot currency exchange rate, x_0; the risk-free interest rate in the foreign currency, r_f; and the current bond price, B_0.

Underlying Asset	Parity Relationship
Futures Contract	$e^{-rT} F_{0,T} = C(K,T) - P(K,T) + e^{-rT} K$
Stock, No-Dividend	$S_0 = C(K,T) - P(K,T) + e^{-rT} K$
Stock, Discrete Dividend	$S_0 - PV_{0,T}(Div) = C(K,T) - P(K,T) + e^{-rT} K$
Stock, Continuous Dividend	$e^{-\delta T} S_0 = C(K,T) - P(K,T) + e^{-rT} K$
Currency	$e^{-r_f T} x_0 = C(K,T) - P(K,T) + e^{-rT} K$
Bond	$B_0 - PV_{0,T}(Coupons) = C(K,T) - P(K,T) + e^{-rT} K$

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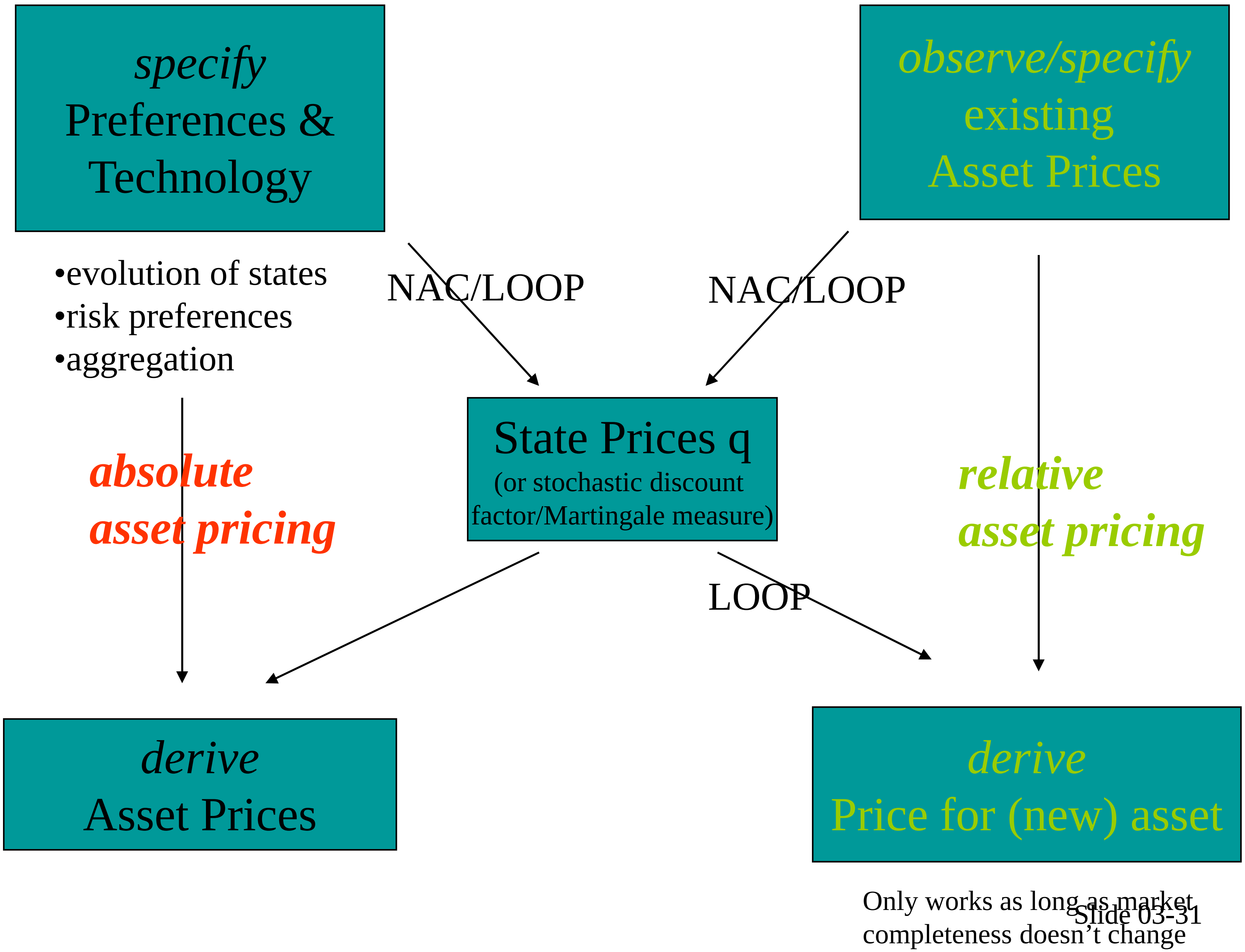
… back to the big picture

- State space (evolution of states)
- (Risk) preferences
- Aggregation over different agents
- Security structure – prices of traded securities
- *Problem:*
  - *Difficult to observe risk preferences*
  - *What can we say about* ***existence of state prices*** *without assuming specific utility functions/constraints for all agents in the economy*


specify
Preferences & Technology
•evolution of states
•risk preferences
•aggregation
observe/specify
existing
Asset Prices
NAC/LOOP
NAC/LOOP
State Prices q
(or stochastic discount factor/Martingale measure)
absolute
asset pricing
relative
asset pricing
LOOP
derive
Asset Prices
derive
Price for (new) asset
Only works as long as market completeness doesn't change

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   If $h'X = k'X$ then $p \cdot h = p \cdot k$.
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   nor portfolio k with $k' X > 0$ and $0 \geq p \cdot k$

Pricing

- Define for each $z \in$ <X>,

$$q(z) := \{p \cdot h : z = h'X\}$$

- If LOOP holds *q(z)* is a single-valued and linear functional. (i.e. if h' and h' lead to same z, then price has to be the same)
- Conversely, if *q* is a linear functional defined in <X> then the law of one price holds.

Pricing

- LOOP $\Rightarrow q(h'X) = p \cdot h$
- A linear functional Q in R^S is a valuation function if

  $Q(z) = q(z)$ for each $z \in \langle X \rangle$.
- $Q(z) = q \cdot z$ for some $q \in R^S$, where $q^s = Q(e_s)$, and e_s is the vector with $e_s^s = 1$ and $e_s^i = 0$ if $i \neq s$
  - e_s is an Arrow-Debreu security
- q is a vector of state prices

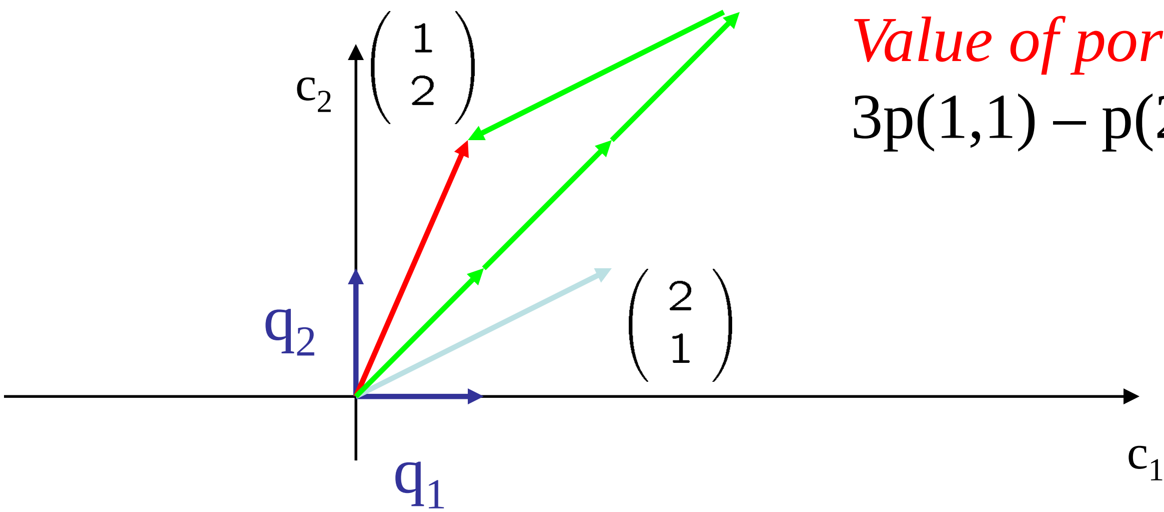
State prices q

- q is a vector of state prices if $p = X\,q$,
  that is $p^j = x^j \leq q$ for each $j = 1,\ldots,J$
- If $Q(z) = q \leq z$ is a valuation functional then q is a vector of state prices
- Suppose q is a vector of state prices and LOOP holds.
  Then if $z = h'X$ LOOP implies that

$$q(z) = \sum_j h^j p^j = \sum_j (\sum_s x_s^j q_s) h^j =$$
$$= \sum_s (\sum_j x_s^j h^j) q_s = q \cdot z$$

- $Q(z) = q \leq z$ is a valuation functional
  iff q is a vector of state prices and LOOP holds

State prices q



$p(1,1) = \; q_1 + q_2$

$p(2,1) = 2q_1 + q_2$

Value of portfolio (1,2)

$3p(1,1) – p(2,1) = 3q_1 + 3q_2 - 2q_1 - q_2$

$= q_1 + 2q_2$

The Fundamental Theorem of Finance

- **Proposition 1.** Security prices exclude arbitrage if and only if there exists a valuation functional with $q >> 0$.
- **Proposition 1'.** Let X be an $J \times S$ matrix, and $p \in R^J$. There is no h in R^J satisfying $h \leq p \leq 0$, $h' X \geq 0$ and at least one strict inequality if, and only if, there exists a vector $q \in R^S$ with $q >> 0$ and $p = X q$.

No arbitrage $\Leftrightarrow$ positive state prices

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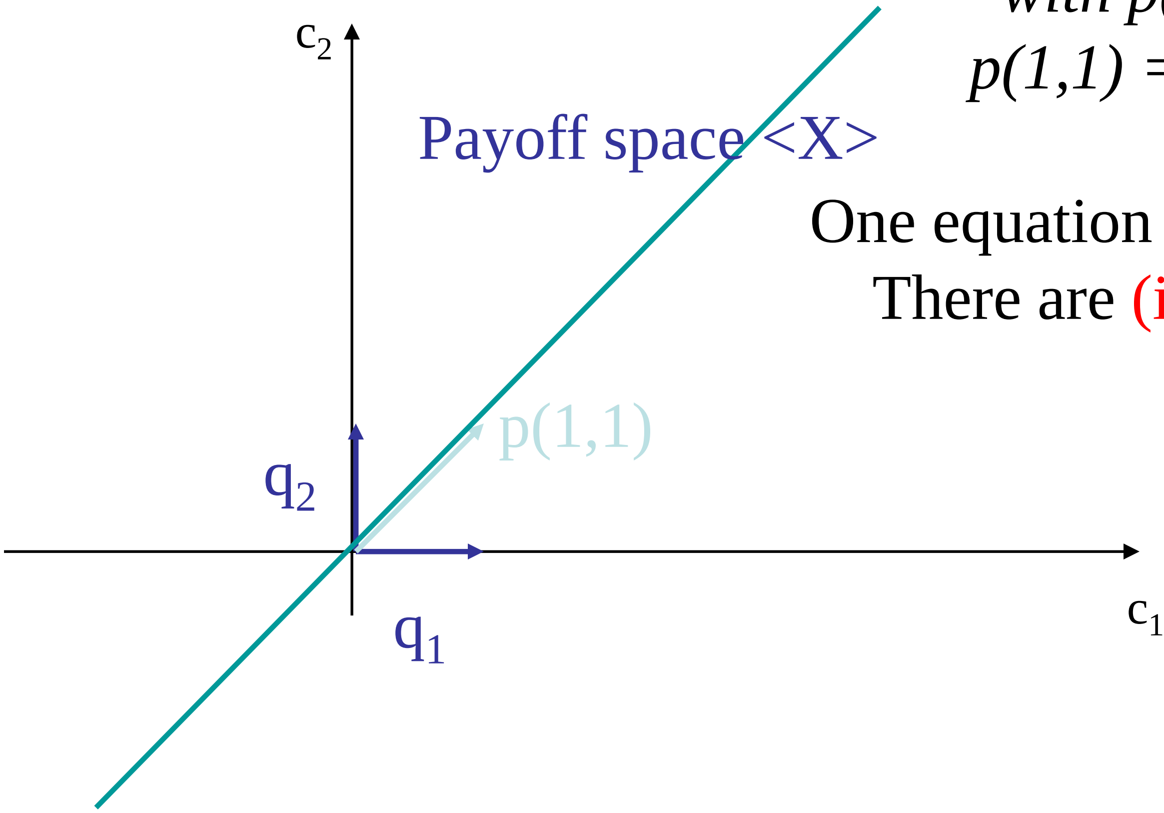
Multiple State Prices q & Incomplete Markets

bond (*1*,*1*) only



What state prices are consistent with p(1,1)?

$p(1,1) = q_1 + q_2$

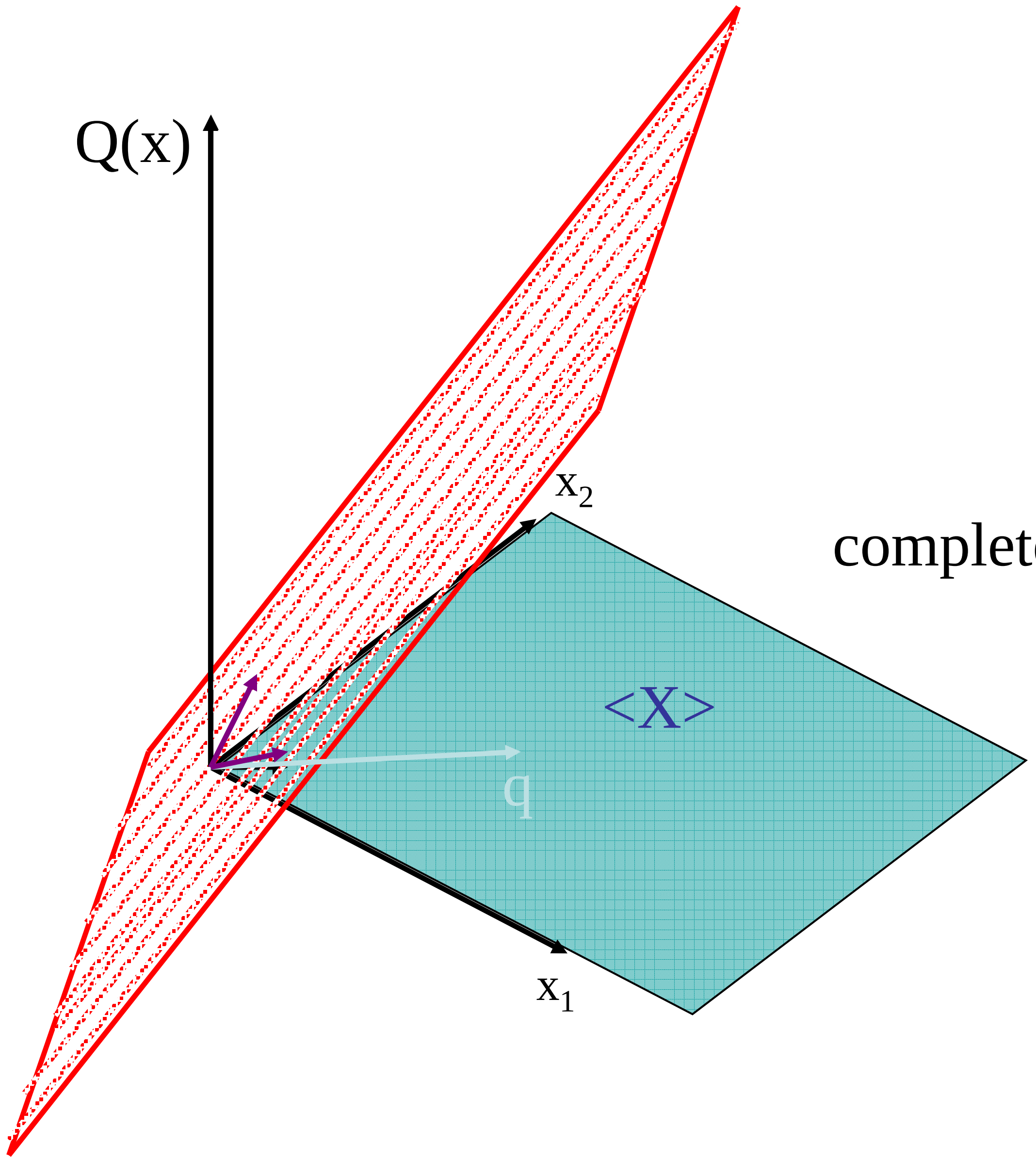
One equation – two unknowns q_1, q_2

There are (infinitely) many.

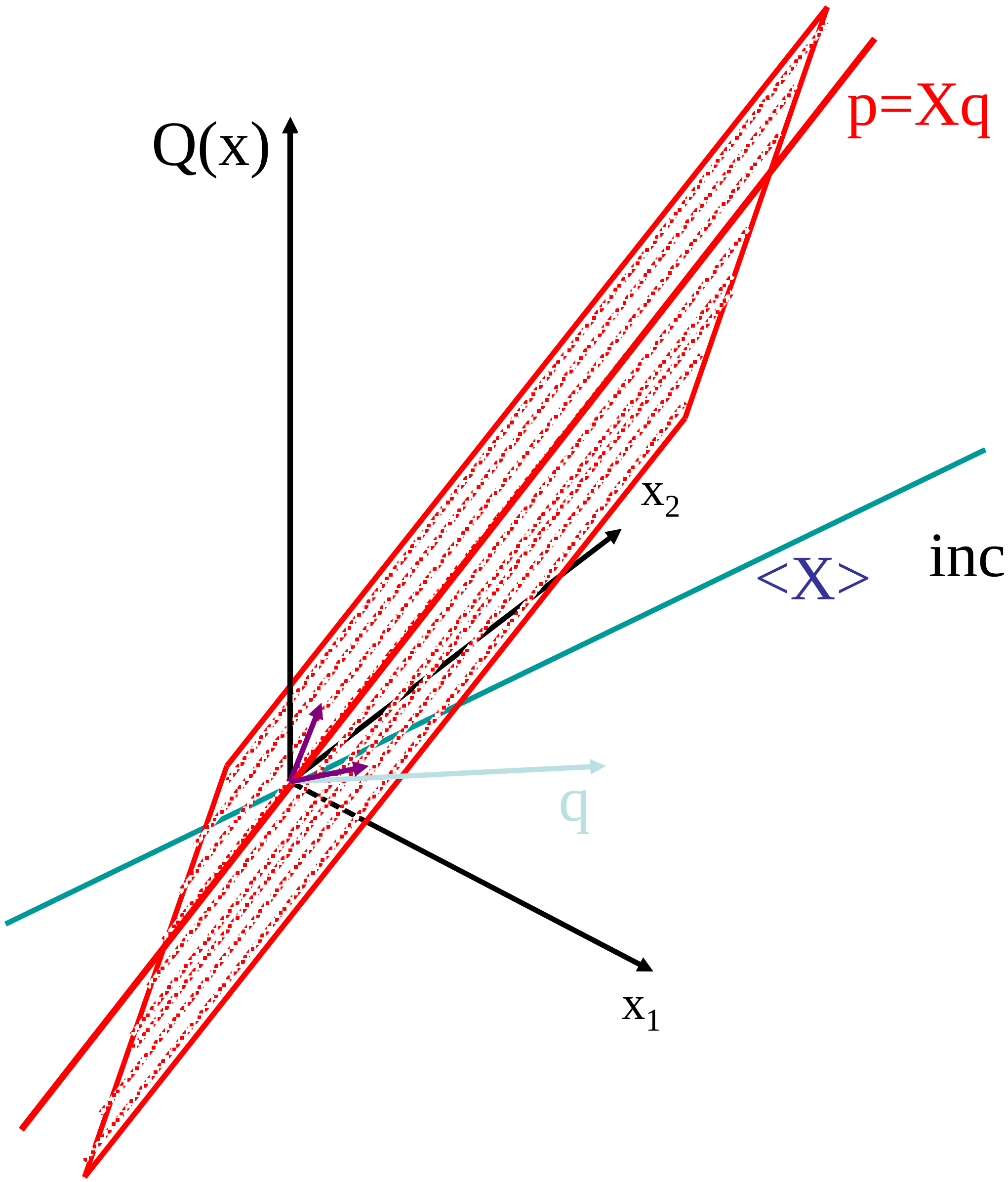
e.g. *if p(1,1)=.9*

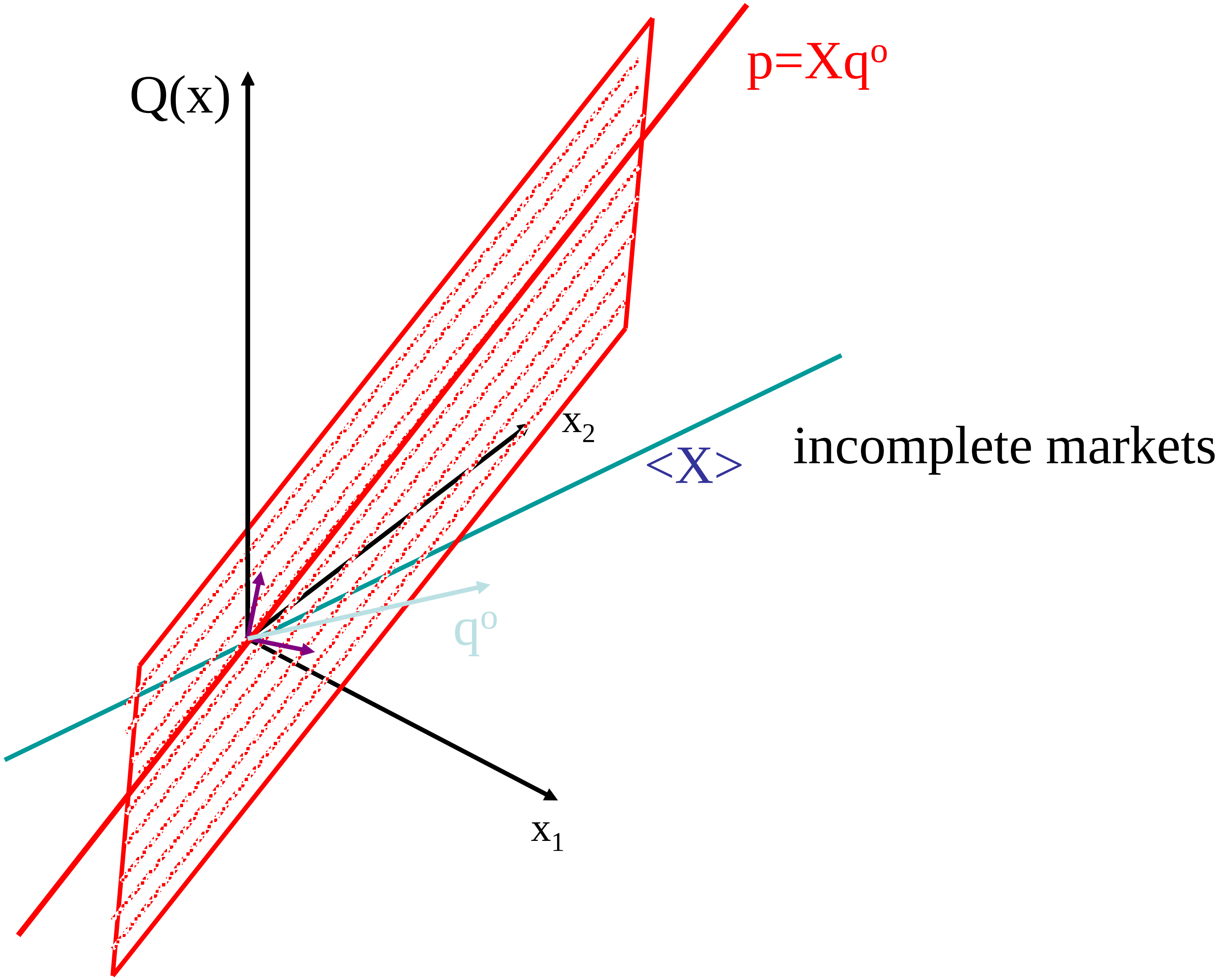
$q_1 = .45$, $q_2 = .45$

or $q_1 = .35$, $q_2 = .55$



complete markets


p=Xq
Q(x)
x2
incomplete markets
<X>
q
x1


p=Xq°
Q(x)
x_2
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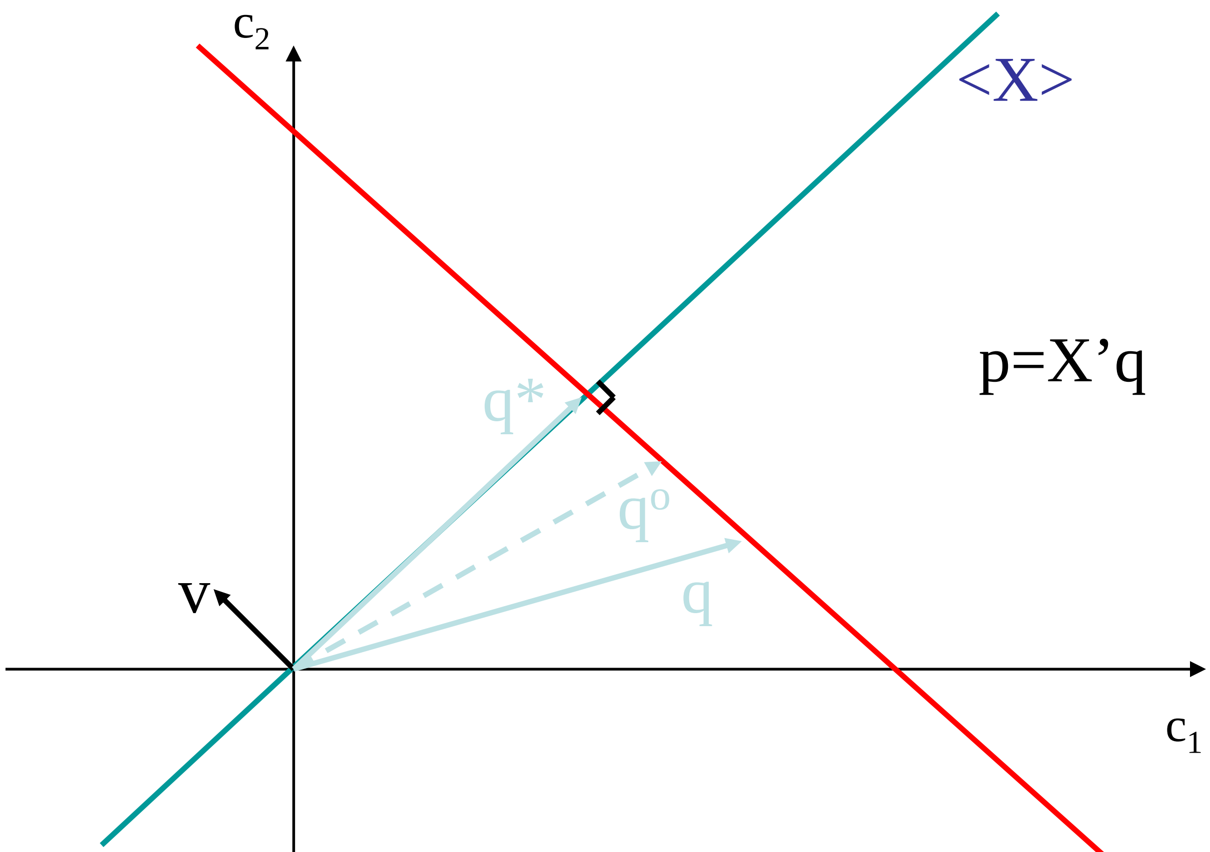
incomplete markets
q°
x_1

Multiple q in incomplete markets



Many possible state price vectors s.t. *p=X'q*.
One is special: q* - it can be replicated as a portfolio.

Uniqueness and Completeness

- **Proposition 2.** If markets are complete, under no arbitrage there exists a *unique* valuation functional.

- If markets are not complete, then there exists
  $v \rangle\backslash R^S$ with $0 = X'v$.
  Suppose there is no arbitrage and let $q >> 0$ be a vector of state prices. Then $q + \alpha\, v >> 0$ provided α is small enough, and $p = X' (q + \alpha\, v)$. Hence, there are an infinite number of strictly positive state prices.

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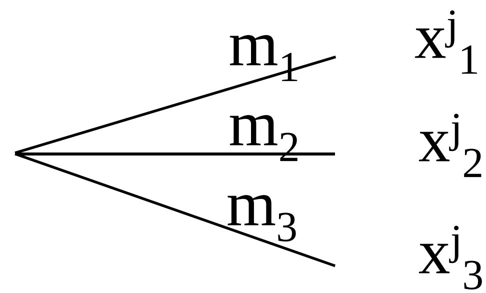
Four Asset Pricing Formulas

1. State prices $p^j = \sum_s q_s\, x_s^j$
2. Stochastic discount factor $p^j = E[mx^j]$



3. Martingale measure $p^j = 1/(1+r^f)\, E_{\hat{\pi}}[x^j]$

   (reflect risk aversion by over(under)weighing the "bad(good)" states!)

4. State-price beta model $E[R^j] - R^f = \beta^j E[R^* - R^f]$

   (in returns $R^j := x^j / p^j$)

1. State Price Model

- … so far price in terms of Arrow-Debreu (state) prices

$$p^j = \sum_s q_s x_s^j$$

2. Stochastic Discount Factor

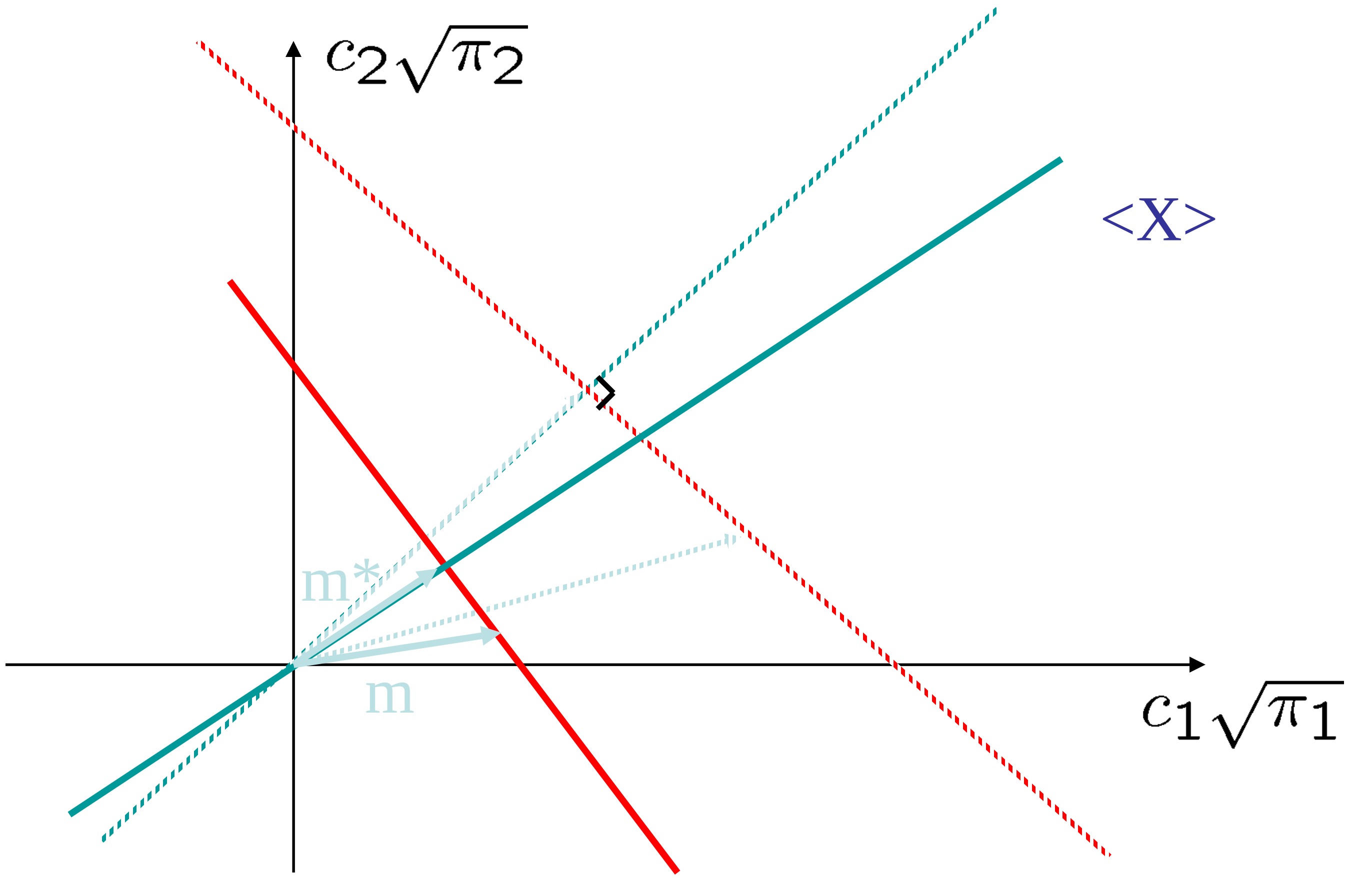
$$p^j = \sum_s q_s x_s^j = \sum_s \pi_s \underbrace{\frac{q_s}{\pi_s}}_{m_s} x_s^j$$

- That is, stochastic discount factor $m_s = q_s/\pi_s$ for all s.

$$p^j = E[mx^j]$$

2. Stochastic Discount Factor

shrink axes by factor $\sqrt{\pi_s}$



Risk-adjustment in payoffs

$p = E[mx^j] = E[m]E[x] + Cov[m,x]$

Since $p^{bond}=E[m1]$. Hence, the risk free rate $R^f = 1/E[m]$.

$$\mathbf{p = E[x]/R^f + Cov[m,x]}$$

Remarks:

(i) If risk-free rate does not exist, R^f is the shadow risk free rate

(ii) Typically $Cov[m,x] < 0$, which lowers price and increases return

3. Equivalent Martingale Measure

- Price of any asset $p^j = \sum_s q_s x_s^j$
- Price of a bond $p^{\text{bond}} = \sum_s q_s = \frac{1}{1+r^f}$

$$p^j = \sum_{s'} q_{s'} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j$$

$$p^j = \frac{1}{1+r^f} \sum_s \underbrace{\frac{q_s}{\sum_{s'} q_{s'}}}_{:=\widehat{\pi}_s} x_s^j$$

$$p^j = \frac{1}{1+r^f} E_{\widehat{\pi}}[x^j]$$

… in Returns: $R^j = x^j/p^j$

$E[mR^j]=1$ $\qquad$ $R^f\, E[m]=1$

ℑ⊤ $E[m(R^j-R^f)]=0$

$E[m]\{E[R^j]-R^f\} + Cov[m,R^j]=0$

$$\mathbf{E[R^j] - R^f = - Cov[m,R^j]/E[m]} \qquad \mathbf{(2)}$$

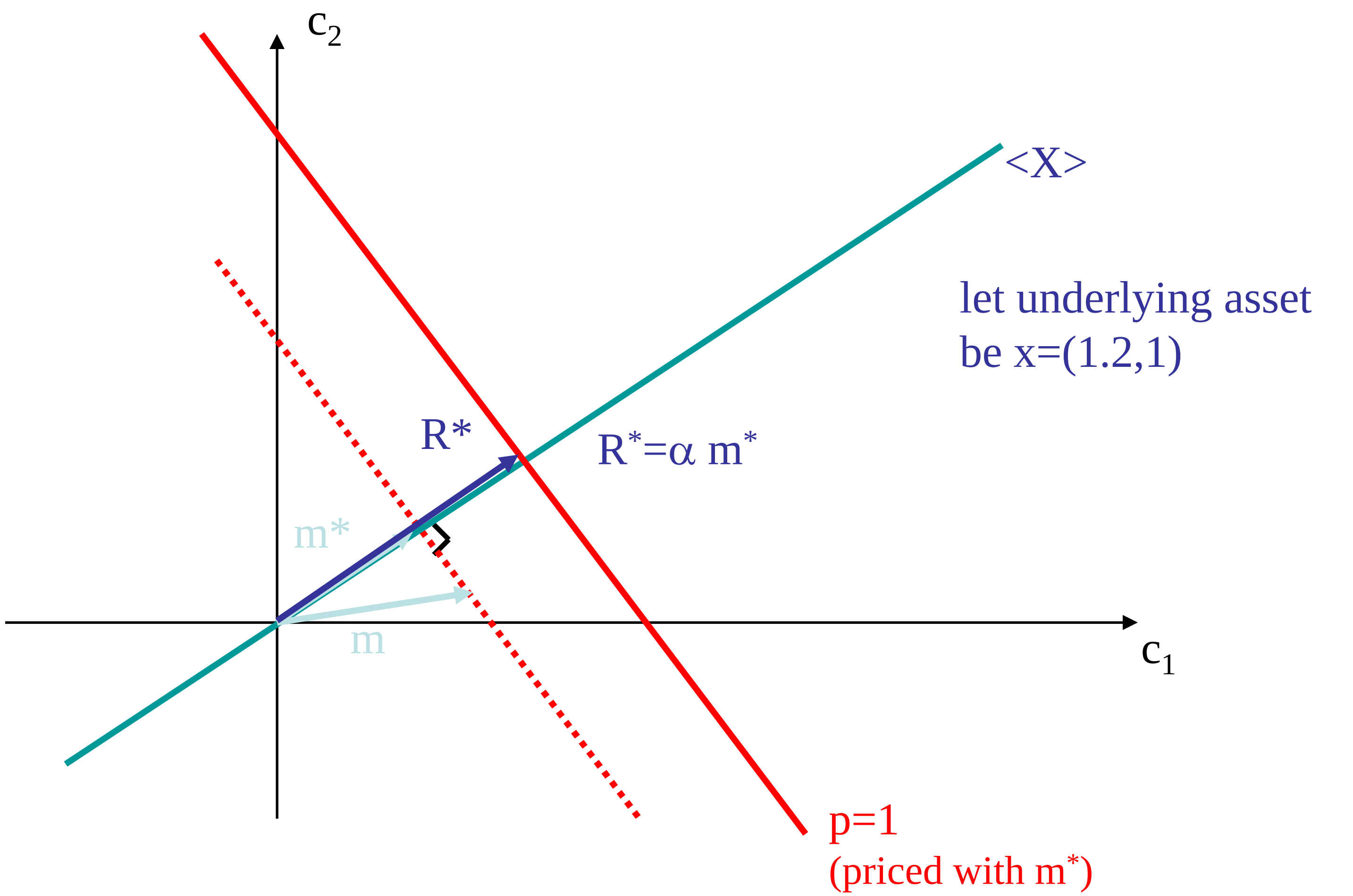
also holds for portfolios *h*

Note:

- risk correction depends only on Cov of payoff/return with discount factor.
- Only compensated for taking on systematic risk not idiosyncratic risk.

4. State-price BETA Model

shrink axes by factor $\sqrt{\pi_s}$



4. State-price BETA Model

$E[R^j] - R^f = - Cov[m,R^j]/E[m]$ (2)

also holds for all portfolios *h* and

we can replace m with m^*

Suppose (i) $Var[m^*] > 0$ and (ii) $R^* = \alpha m^*$ with $\alpha > 0$

$E[R^h] - R^f = - Cov[R^*,R^h]/E[R^*]$ (2')

Define β^h **$:= Cov[R^*,R^h]/ Var[R^*]$** for any portfolio *h*

4. State-price BETA Model

(2) for R^h: $E[R^h]-R^f=-Cov[R^*,R^h]/E[R^*]$

$= -\beta^h \; Var[R^*]/E[R^*]$

(2) for R^*: $E[R^*]-R^f=-Cov[R^*,R^*]/E[R^*]$

$=-Var[R^*]/E[R^*]$

Hence,

$$E[R^h] - R^f = \beta^h \, E[R^* - R^f]$$

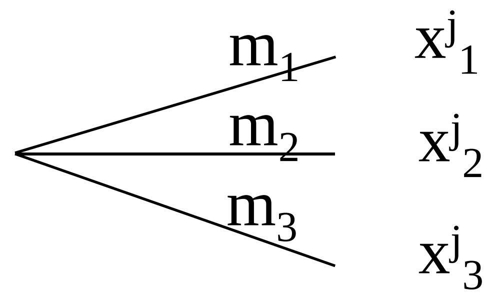
where $\beta^h := Cov[R^*,R^h]/Var[R^*]$

very general – but what is R^* in reality?

Regression $R^h_s = \alpha^h + \beta^h (R^*)_s + \varepsilon_s$ with $Cov[R^*,\varepsilon]=E[\varepsilon]=0$

Four Asset Pricing Formulas

1. State prices $1 = \sum_s q_s R_s^j$
2. Stochastic discount factor $1 = E[mR^j]$



3. Martingale measure $1 = 1/(1+r^f)\, E_{\hat{\pi}}[R^j]$
   (reflect risk aversion by
   over(under)weighing the "bad(good)" states!)
4. State-price beta model $E[R^j] - R^f = \beta^j\, E[R^* - R^f]$
   (in returns $R^j := x^j / p^j$)

What do we know about q, m, π, R*?

- Main results so far
  - Existence iff no arbitrage
    - ⇨ Hence, single factor only
    - but doesn't famos Fama-French factor model has 3 factors?
    - ⇨ multiple factor is due to time-variation (wait for multi-period model)
  - Uniqueness if markets are complete

Different Asset Pricing Models

$p_t = E[m_{t+1} x_{t+1}]$

where $m_{t+1}=f(\leq, \ldots, \leq)$

f(·) = asset pricing model

General Equilibrium

$f(\leq) = MRS / \pi$

Factor Pricing Model

$a+b_1 f_{1,t+1} + b_2 f_{2,t+1}$

CAPM

$a+b_1 f_{1,t+1} = a+b_1 R^M$

$\Im\top$ $E[R^h] - R^f = \beta^h E[R^* - R^f]$

where $\beta^h := Cov[R^*, R^h]/Var[R^*]$

CAPM

$R^* = R^f (a+b_1 R^M)/(a+b_1 R^f)$

where R^M = return of market portfolio

Is $b_1 < 0$?

Different Asset Pricing Models

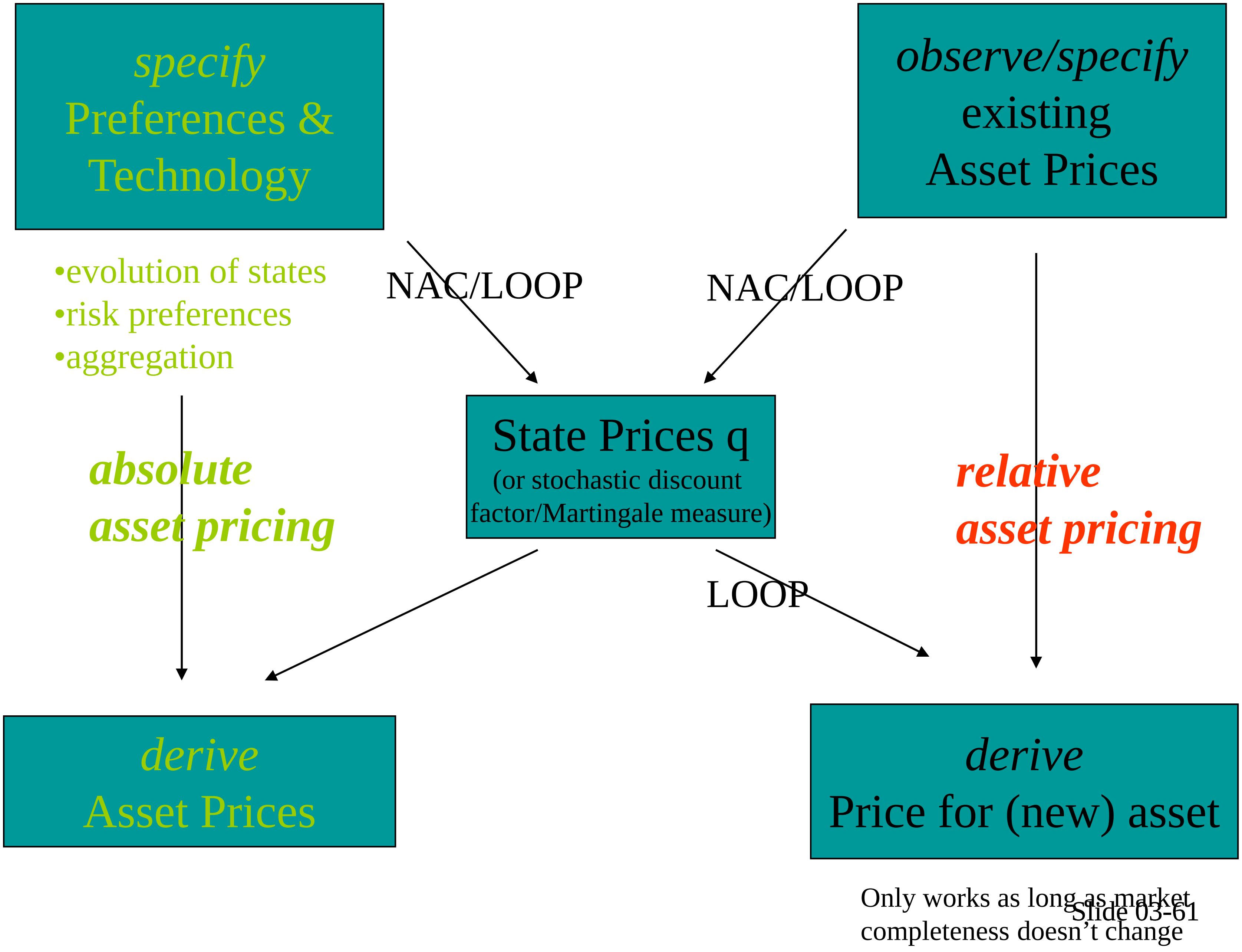
- Theory
  - All economics and modeling is determined by

$$m_{t+1} = a + \mathbf{b' f}$$

  - Entire content of model lies in restriction of SDF
- Empirics
  - m^* (which is a portfolio payoff) prices as well as m (which is e.g. a function of income, investment etc.)
  - measurement error of m^* is smaller than for any m
  - Run regression on *returns* (portfolio payoffs)! (e.g. Fama-French three factor model)

Overview: Pricing - one period model

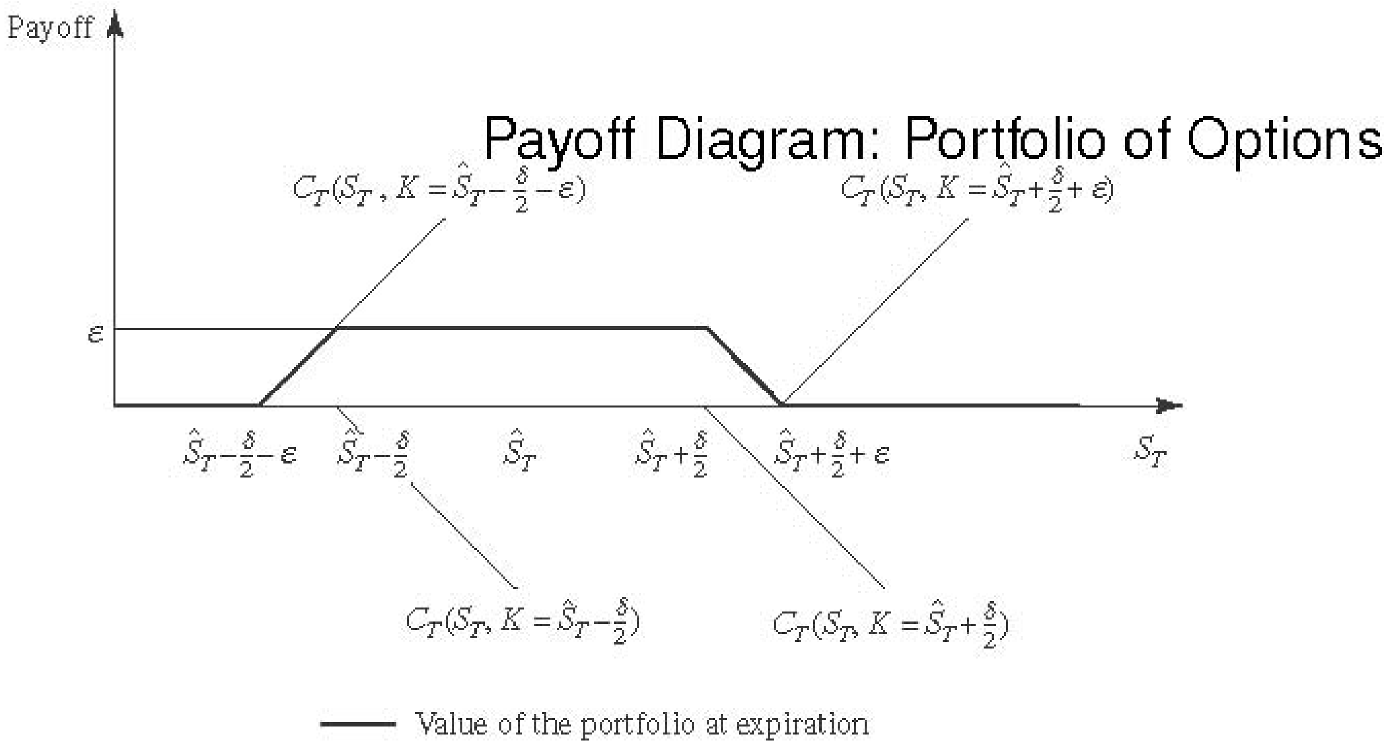
1. LOOP, No arbitrage
2. Forwards
3. Options: Parity relationship
4. No arbitrage and existence of state prices
5. Market completeness and uniqueness of state prices
6. Pricing kernel q^*
7. Four pricing formulas:
   state prices, SDF, EMM, beta pricing
8. Recovering state prices from options


specify
Preferences &
Technology
•evolution of states
•risk preferences
•aggregation
observe/specify
existing
Asset Prices
NAC/LOOP
NAC/LOOP
State Prices q
(or stochastic discount
factor/Martingale measure)
absolute
asset pricing
relative
asset pricing
LOOP
derive
Asset Prices
derive
Price for (new) asset
Only works as long as market
completeness doesn't change

Recovering State Prices from Option Prices

- Suppose that S_T, the price of the underlying portfolio (we may think of it as a proxy for price of "market portfolio"), assumes a "continuum" of possible values.
- Suppose there are a "continuum" of call options with different strike/exercise prices $\Rightarrow$ markets are complete
- Let us construct the following portfolio:
  for some small positive number $\varepsilon>0$,
  - Buy one call with $E = \hat{S}_T - \frac{\delta}{2} - \varepsilon$
  - Sell one call with $E = \hat{S}_T - \frac{\delta}{2}$
  - Sell one call with $E = \hat{S}_T + \frac{\delta}{2}$
  - Buy one call with $E = \hat{S}_T + \frac{\delta}{2} + \varepsilon$

Recovering State Prices … (ctd.)



Recovering State Prices … (ctd.)

- Let us thus consider buying $^1/_\varepsilon$ units of the portfolio. The total payment, when $\hat{S}_T - \frac{\delta}{2} \leq S_T \leq \hat{S}_T + \frac{\delta}{2}$, is $\varepsilon \cdot \frac{1}{\varepsilon} \equiv 1$, for any choice of ε. We want to let $\varepsilon \mapsto 0$, so as to eliminate the payments in the ranges $S_T \in [\hat{S}_T - \frac{\delta}{2} - \varepsilon, \hat{S}_T - \frac{\delta}{2})$ and $S_T \in (\hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \varepsilon]$. The value of $^1/_\varepsilon$ units of this portfolio is :

$$\frac{1}{\varepsilon}\{C(S, K = \hat{S}_T - \delta/2 - \varepsilon) - C(S, K = \hat{S}_T - \delta/2)$$

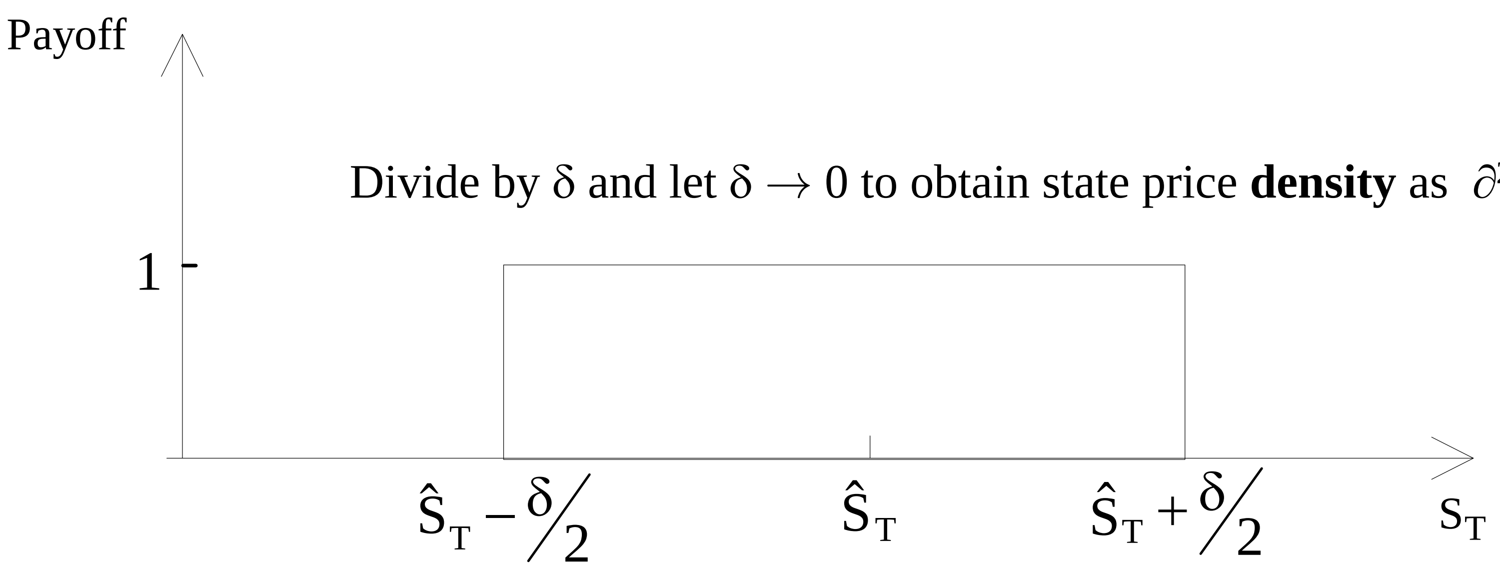
$$-[C(S, K = \hat{S}_T + \delta/2) - C(S, K = \hat{S}_T + \delta/2 + \varepsilon)]\}$$

Taking the limit $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{C(S,K=\hat{S}_T-\delta/2-\varepsilon)-C(S,K=\hat{S}_T-\delta/2)-[C(S,K=\hat{S}_T+\delta/2)-C(S,K=\hat{S}_T+\delta/2+\varepsilon)]\}$$

$$-\lim_{\varepsilon\to 0}\{\underbrace{\frac{C(S,K=\hat{S}_T-\delta/2-\varepsilon)-C(S,K=\hat{S}_T-\delta/2)}{-\varepsilon}}_{\leq 0}\}+\lim_{\varepsilon\to 0}\{\underbrace{\frac{C(S,K=\hat{S}_T+\delta/2+\varepsilon)-C(S,K=\hat{S}_T+\delta/2)}{-\varepsilon}}_{\leq 0}\}$$

$$=-\frac{\partial C}{\partial K}(S,K=\widehat{S}_T-\tfrac{\delta}{2})+\frac{\partial C}{\partial K}(S,K=\widehat{S}_T+\tfrac{\delta}{2})$$



Divide by δ and let $\delta \to 0$ to obtain state price **density** as $\partial^2 C/\partial K^2$.

Recovering State Prices … (ctd.)

Evaluating following cash flow

$$CF_T = \left\{ \begin{array}{ll} 0 & \text{if } S_T \notin [\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}] \\ 50000 & \text{if } S_T \in [\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}] \end{array} \right\}$$

The value today of this cash flow is :

$$50000[\frac{\partial C}{\partial K}(S, K = \widehat{S}_T + \frac{\delta}{2}) - \frac{\partial C}{\partial K}(S, K = \widehat{S}_T - \frac{\delta}{2})]$$

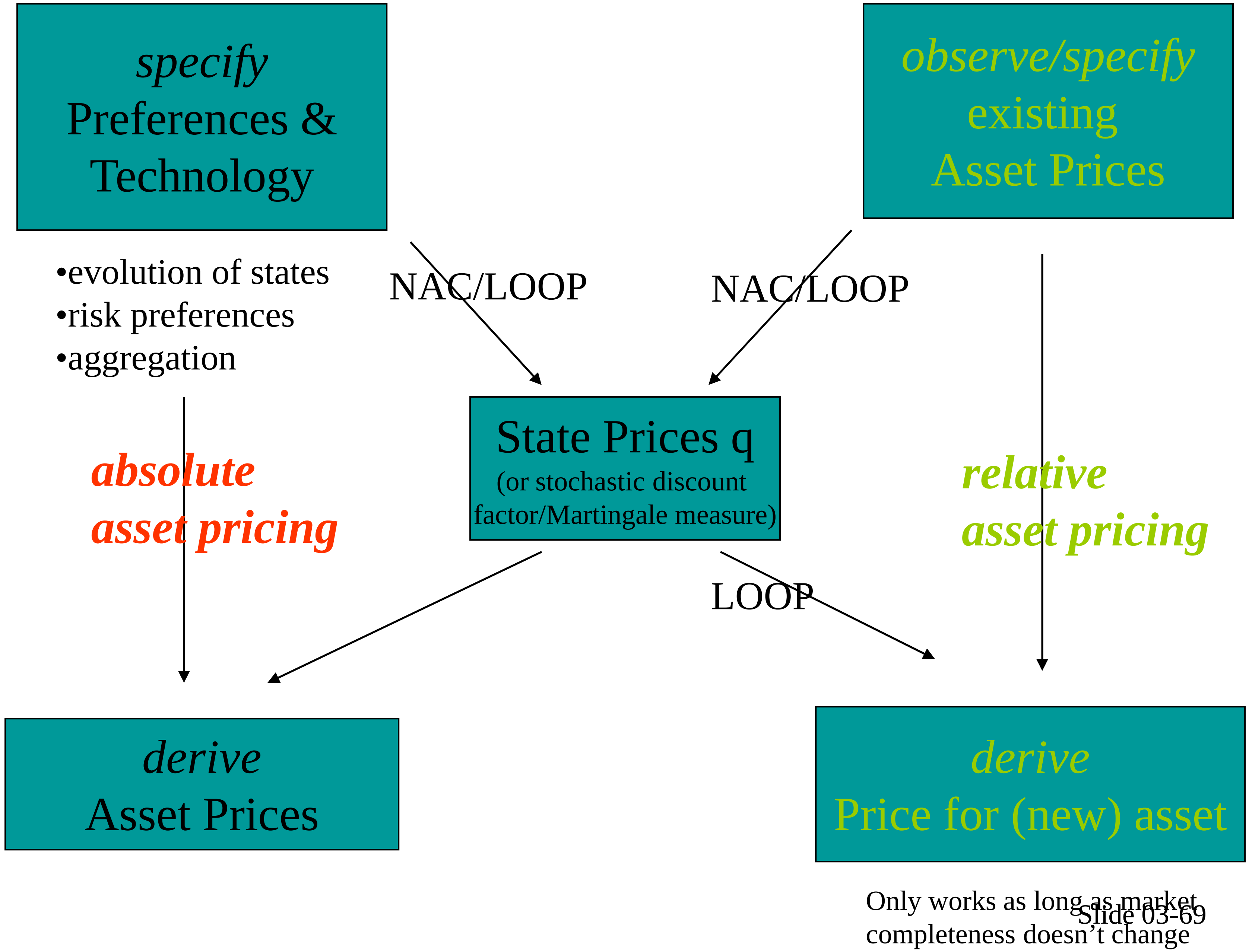
$$q(S_T^1, S_T^2) = \frac{\partial C}{\partial K}(S, K = S_T^2) - \frac{\partial C}{\partial K}(S, K = S_T^1)$$

Recovering State Prices (discrete setting)

			Payoff if $S_T =$								
K	$C(S,K)$	Cost of Position	7	8	9	10	11	12	13	ΔC	$\Delta(\Delta C) = q_\theta$
7	3.354										
										-0.895	
8	2.459										0.106
										-0.789	
9	1.670	+1.670	0	0	0	1	2	3	4		0.164
										-0.625	
10	1.045	-2.090	0	0	0	0	-2	-4	-6		**0.184**
										-0.441	
11	0.604	+0.604	0	0	0	0	0	1	2		0.162
										-0.279	
12	0.325										0.118
										-0.161	
13	0.164										
		0.184	0	0	0	1	0	0	0		

Table 8.1 Pricing an Arrow-Debreu State Claim

E	C(S,E)	Cost of position	Payoff if S_T =								
			7	8	9	10	11	12	13	ΔC	$\Delta(\Delta C) = q_s$
7	3.354										
										-0.895	
8	2.459										0.106
										-0.789	
9	1.670	+1.670	0	0	0	1	2	3	4		0.164
										-0.625	
10	1.045	-2.090	0	0	0	0	-2	-4	-6		**0.184**
										-0.441	
11	0.604	+0.604	0	0	0	0	0	1	2		0.162
										-0.279	
12	0.325										0.118
										-0.161	
13	0.164										
		0.184	0	0	0	1	0	0	0		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derive
Asset Prices
derive
Price for (new) asset
Only works as long as market
completeness doesn't change