# *Lecture 04: Risk Preferences and Expected Utility Theory*

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# Overview: Risk Preferences

1. State-by-state dominance
2. Stochastic dominance [DD4]
3. vNM expected utility theory
   a) Intuition [L4]
   b) Axiomatic foundations [DD3]
4. Risk aversion coefficients and portfolio choice [DD5,L4]
5. Prudence coefficient and precautionary savings [DD5]
6. Mean-variance preferences [L4.6]

# State-by-state Dominance

- State-by-state dominance ⇒ incomplete ranking
- « riskier »

Table 2.1 Asset Payoffs ($)

| | t = 0 | t = 1 | |
|---|---|---|---|
| | Cost at t=0 | Value at t=1 $\pi_1 = \pi_2 = ½$ | |
| | | s = 1 | s = 2 |
| investment 1 | - 1000 | 1050 | 1200 |
| investment 2 | - 1000 | 500 | 1600 |
| investment 3 | - 1000 | 1050 | 1600 |

- investment 3 state by state dominates 1.

# State-by-state Dominance (ctd.)

Table 2.2 State Contingent ROR (r )

| | State Contingent ROR (r ) | | | |
|---|---|---|---|---|
| | s = 1 | s = 2 | Er | σ |
| Investment 1 | 5% | 20% | 12.5% | 7.5% |
| Investment 2 | -50% | 60% | 5% | 55% |
| Investment 3 | 5% | 60% | 32.5% | 27.5% |

- Investment 1 mean-variance dominates 2
- BUT investment 3 does not m-v dominate 1!

# State-by-state Dominance (ctd.)

Table 2.3 State Contingent Rates of Return

| | State Contingent Rates of Return | |
|---|---|---|
| | s = 1 | s = 2 |
| investment 4 | 3% | 5% |
| investment 5 | 3% | 8% |
| | $\pi_1 = \pi_2 = ½$<br>$E[r_4] = 4\%$; $\sigma_4 = 1\%$<br>$E[r_5] = 5.5\%$; $\sigma_5 = 2.5\%$ | |

- What is the trade-off between risk and expected return?
- Investment 4 has a higher Sharpe ratio $(E[r]-r^f)/\sigma$ than investment 5 for $r^f = 0$.

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# Stochastic Dominance

- Still incomplete ordering
  - "More complete" than state-by-state ordering
  - State-by-state dominance $\Rightarrow$ stochastic dominance
  - Risk preference not needed for ranking!
    - independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. ("risk-preference-free")
- Next Section:
  - Complete preference ordering and utility representations

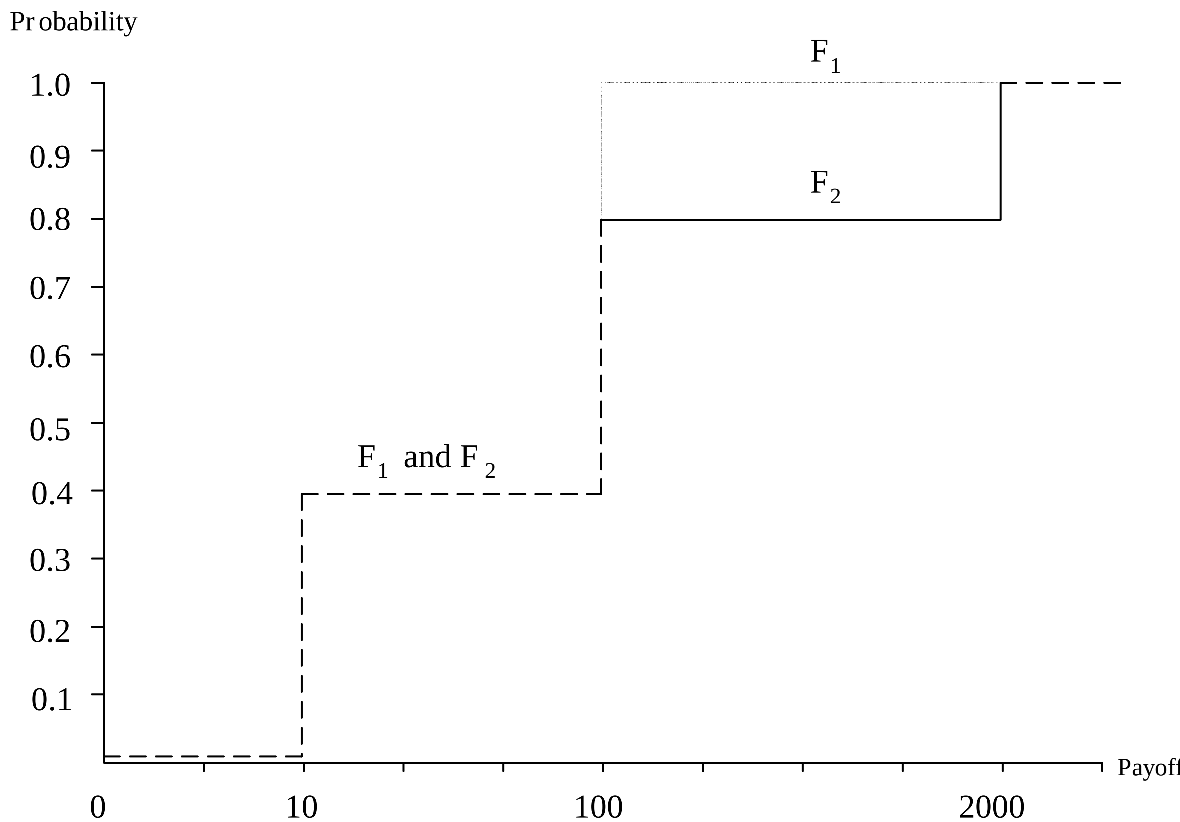
*Homework:* Provide an example which can be ranked according to FSD , but not according to state dominance.

# Table 3-1 Sample Investment Alternatives

| States of nature | 1 | 2 | 3 |
|---|---|---|---|
| Payoffs | 10 | 100 | 2000 |
| Proba $Z_1$ | .4 | .6 | 0 |
| Proba $Z_2$ | .4 | .4 | .2 |

$EZ_1 = 64,\ \sigma_{z_1} = 44$

$EZ_2 = 444,\ \sigma_{z_2} = 779$



# First Order Stochastic Dominance

- <u>Definition 3.1</u> : *Let* $F_A(x)$ *and* $F_B(x)$ , *respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval [a,b]. We say that* $F_A(x)$ *first order stochastically dominates (FSD)* $F_B(x)$ *if and only if for all* $x \in [a,b]$

$$F_A(x) \leq F_B(x)$$

*Homework:* Provide an example which can be ranked according to FSD , but not according to state dominance.

# First Order Stochastic Dominance

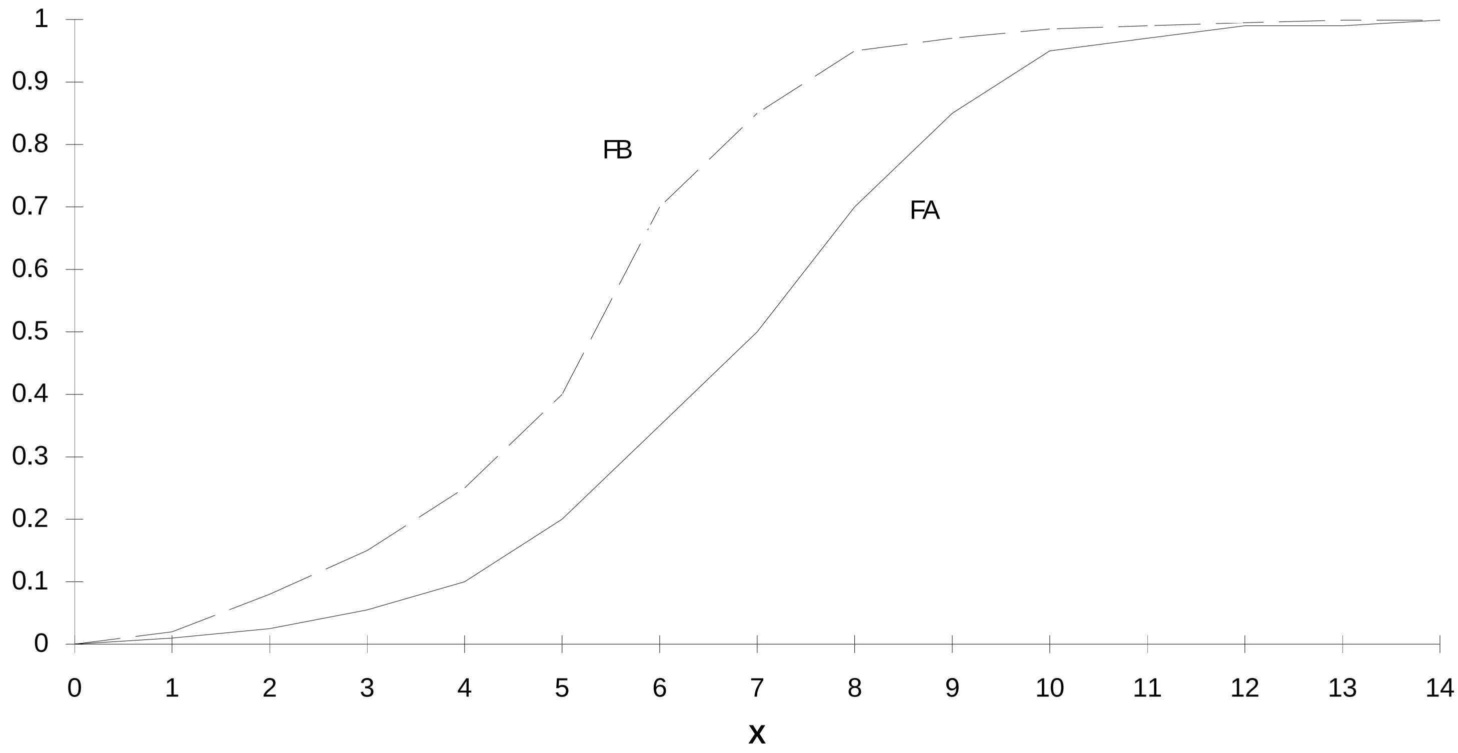


Table 3-2 Two Independent Investments

| Investment 3 | | Investment 4 | |
|---|---|---|---|
| Payoff | Prob. | Payoff | Prob. |
| 4 | 0.25 | 1 | 0.33 |
| 5 | 0.50 | 6 | 0.33 |
| 12 | 0.25 | 8 | 0.33 |

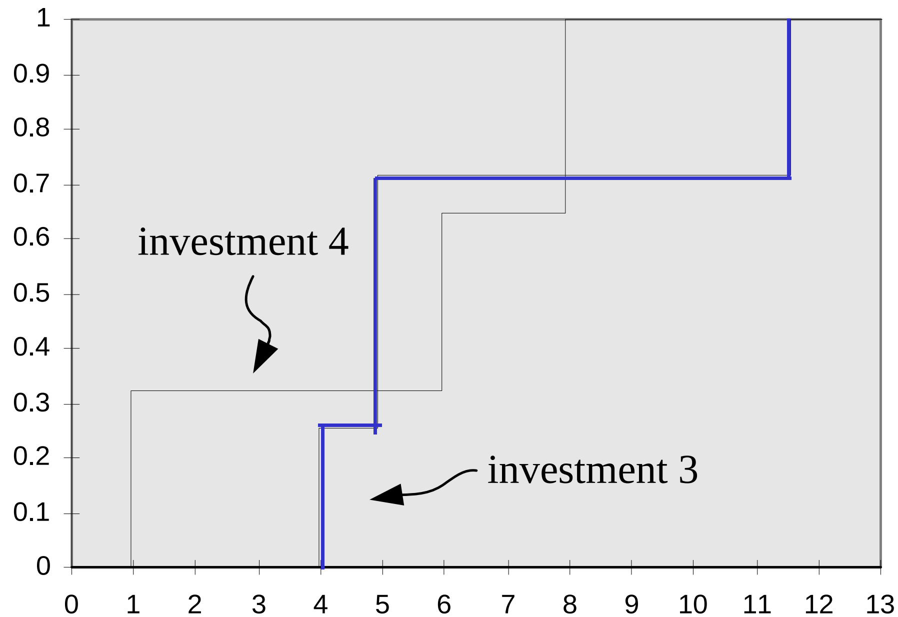



Figure 3-6 Second Order Stochastic Dominance Illustrated

# Second Order Stochastic Dominance

- Definition 3.2: *Let* $F_A(\tilde{x})$, $F_B(\tilde{x})$, *be two cumulative probability distribution for random payoffs in* $[a,b]$. *We say that* $F_A(\tilde{x})$ *second order stochastically dominates* (SSD) $F_B(\tilde{x})$ *if and only if for any* $x$ :

$$\int_{-\infty}^{x} \left[ F_B(t) - F_A(t) \right] dt \geq 0$$

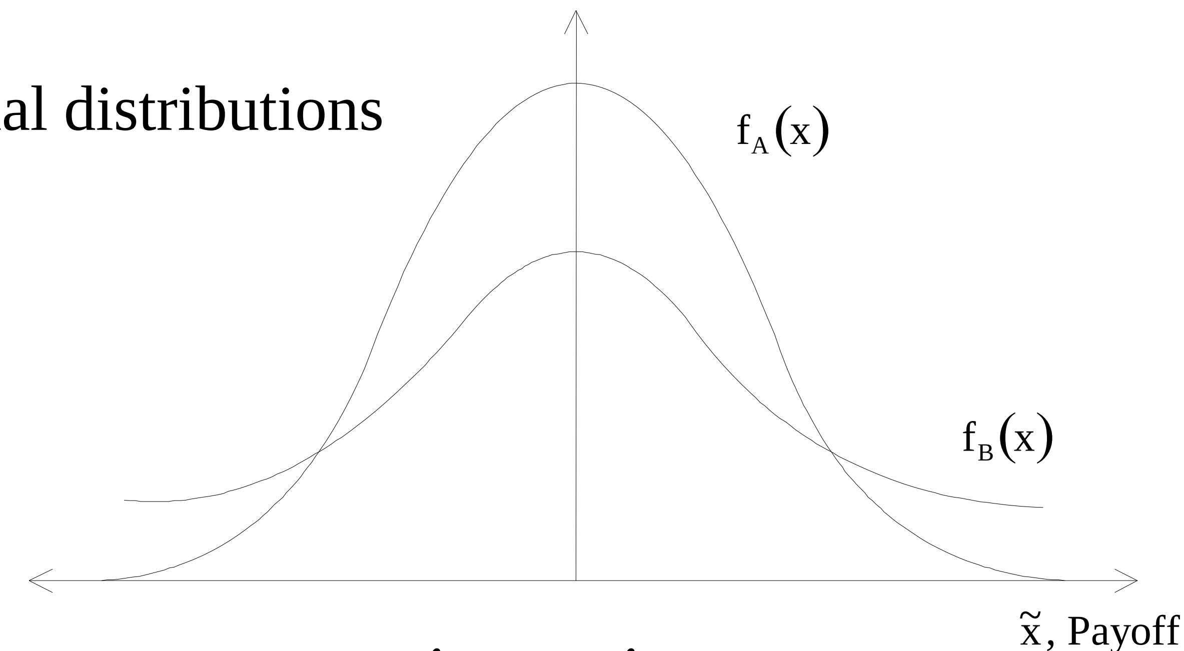
(with strict inequality for some meaningful interval of values of *t*).

# Mean Preserving Spread

$x_B = x_A + z$ (3.8)

where z is independent of $x_A$ and has zero mean

for normal distributions




$$\mu = \int x\, f_A(x) dx = \int x\, f_B(x) dx$$

Figure 3-7 Mean Preserving Spread

# Mean Preserving Spread & SSD

- Theorem 3.4 : *Let* $F_A(\bullet)$ *and* $F_B(\bullet)$ *be two distribution functions defined on the same state space with identical means. Then the follow statements are equivalent* :
  - $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$
  - $F_B(\tilde{x})$ *is a mean preserving spread of* $F_A(\tilde{x})$ *in the sense of Equation* (3.8) *above*.

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# A Hypothetical Gamble

- Suppose someone offers you this gamble:
  - "I have a fair coin here. I'll flip it, and if it's tail I pay you $1 and the gamble is over. If it's head, I'll flip again. If it's tail then, I pay you $2, if not I'll flip again. With every round, I double the amount I will pay to you if it's tail."
- Sounds like a good deal. After all, you can't loose. So here's the question:
  - **How much are you willing to pay to take this gamble?**

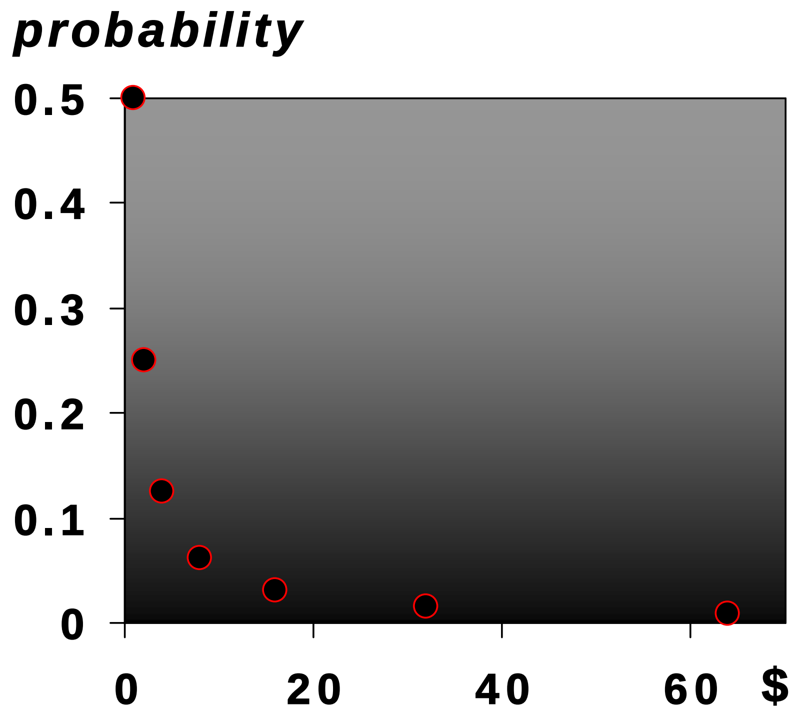
# Proposal 1: Expected Value

- With probability 1/2 you get \$1. $\left(\tfrac{1}{2}\right)^1$ times $2^0$
- With probability 1/4 you get \$2. $\left(\tfrac{1}{2}\right)^2$ times $2^1$
- With probability 1/8 you get \$4. $\left(\tfrac{1}{2}\right)^3$ times $2^2$
- etc.
  - The expected payoff is the sum of these payoffs, weighted with their probabilities, so

$$\sum_{t=1}^{\infty} \underbrace{\left(\frac{1}{2}\right)^t}_{probability} \cdot \underbrace{2^{t-1}}_{payoff} = \sum_{t=1}^{\infty} \frac{1}{2} = \infty$$

# An Infinitely Valuable Gamble?

- You should pay everything you own *and more* to purchase the right to take this gamble!
- Yet, in practice, no one is prepared to pay such a high price. Why?
- Even though the expected payoff is infinite, the distribution of payoffs is not attractive…




- With 93% probability we get $8 or less, with 99% probability we get $64 or less.

# What Should We Do?

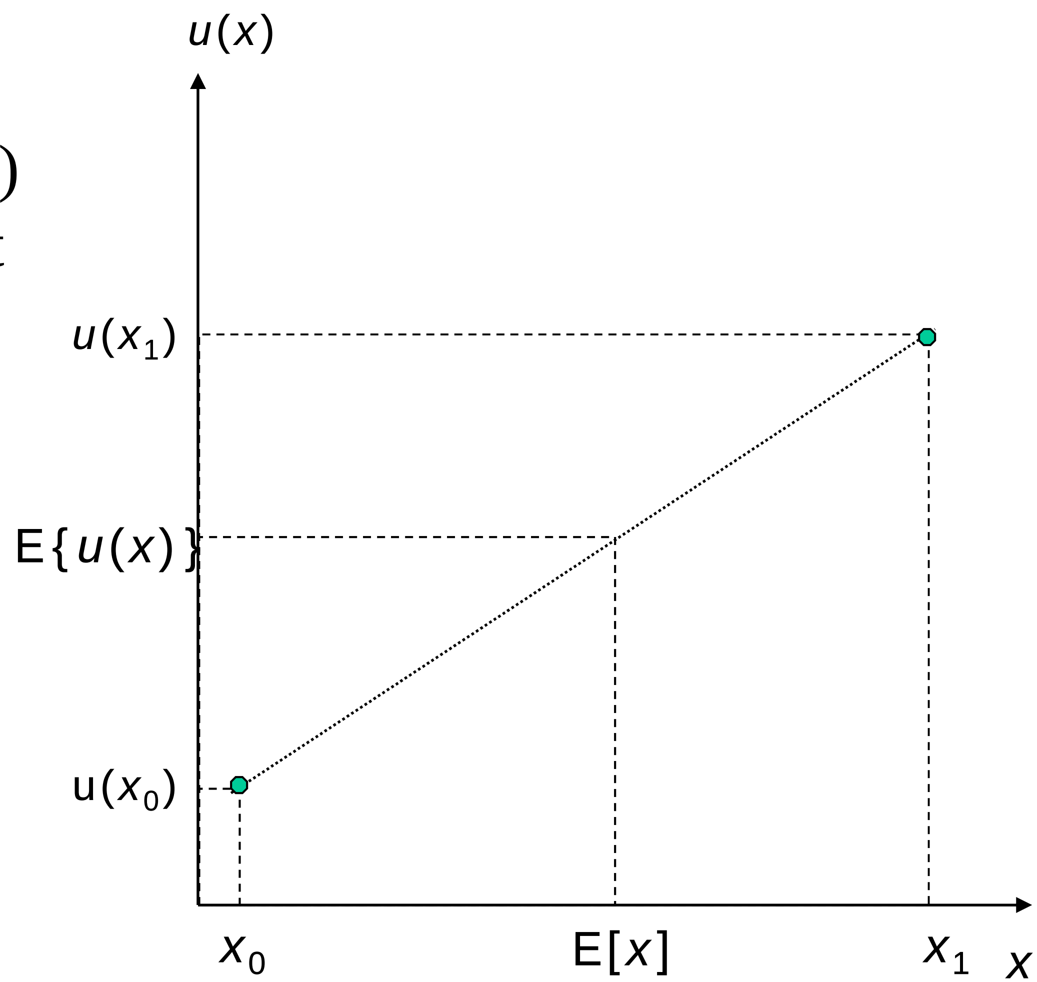
- How can we decide in a rational fashion about such gambles (or investments)?
- Proposal 2: Bernoulli suggests that large gains should be weighted less. He suggests to use the natural logarithm. [Cremer - another great mathematician of the time - suggests the square root.]

$$\sum_{t=1}^{\infty} \underbrace{\left(\frac{1}{2}\right)^t}_{probability} \cdot \underbrace{\ln(2^{t-1})}_{utility\ of\ payoff} = \ln(2) = \begin{matrix}\text{expected utility} \\ \text{of gamble}\end{matrix} < \infty$$

↗ Bernoulli would have paid at most $e^{\ln(2)} = \$2$ to participate in this gamble.

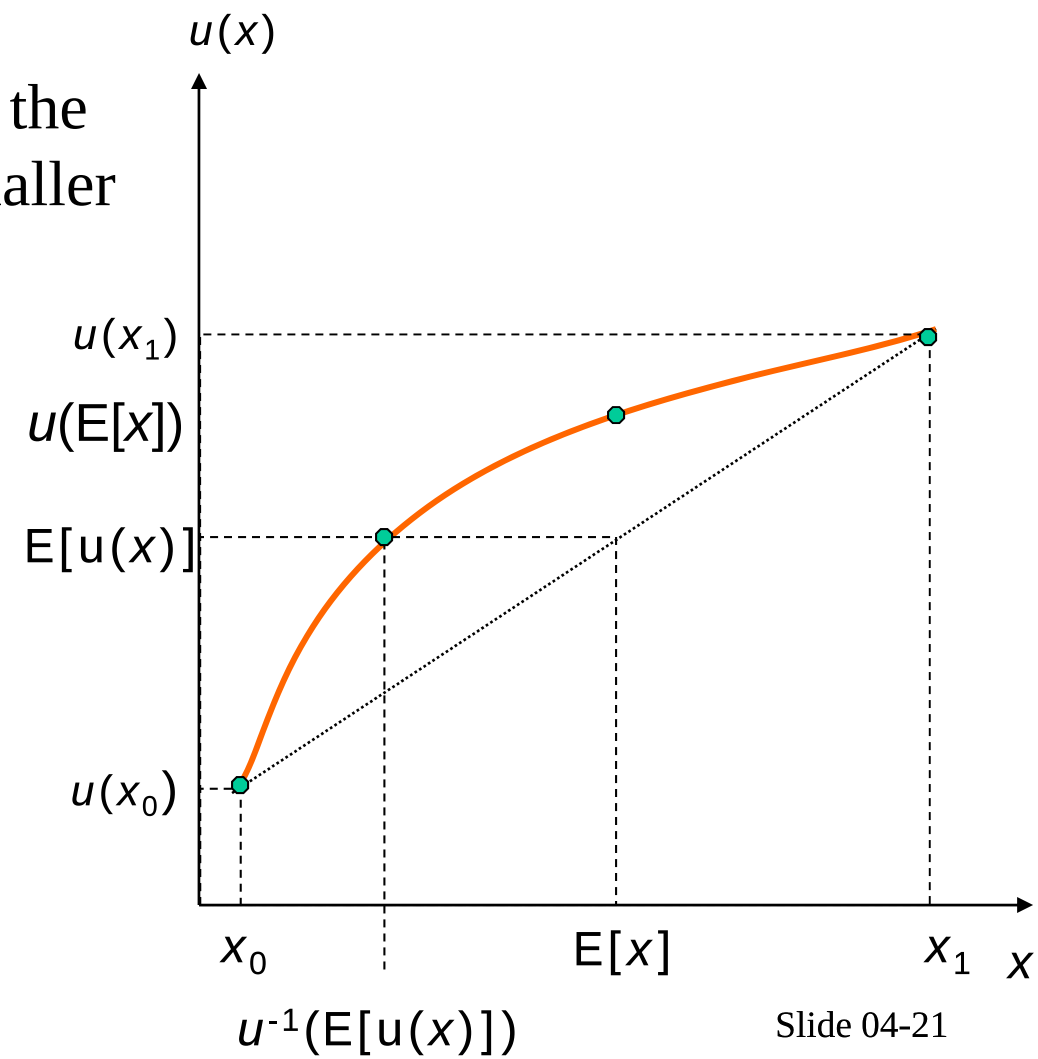
# Risk-Aversion and Concavity

- The shape of the von Neumann Morgenstern (NM) utility function contains a lot of information.
- Consider a fifty-fifty lottery with final wealth of $x_0$ or $x_1$



# Risk-aversion and concavity

- Risk-aversion means that the certainty equivalent is smaller than the expected prize.
  - We conclude that a risk-averse vNM utility function must be concave.



# Jensen's Inequality

**Theorem 3.1 (Jensen's Inequality):**

- *Let* g( ) *be a concave function on the interval* [*a*,*b*], *and be a random variable such that*

  $$\text{Prob}\,(x \in [a,b]) = 1$$

- *Suppose the expectations* E(x) *and* E[g(x)] *exist; then*

  $$E\left[g(\tilde{x})\right] \leq g\left[E(\tilde{x})\right]$$

  *Furthermore, if* g(•) *is strictly concave, then the inequality is strict.*

# Representation of Preferences

A preference ordering is (i) complete, (ii) transitive, (iii) continuous and [(iv) relatively stable] can be represented by a utility function, i.e.

$$(c_0, c_1, \ldots, c_S) \succ (c'_0, c'_1, \ldots, c'_S)$$
$$\Leftrightarrow \; U(c_0, c_1, \ldots, c_S) > U(c'_0, c'_1, \ldots, c'_S)$$

(preference ordering *over lotteries* – ($S$+1)-dimensional space)

# Preferences over Prob. Distributions

- Consider $c_0$ fixed, $c_1$ is a random variable
- Preference ordering over probability distributions
- Let
  - P be a set of probability distributions with a finite support over a set X,
  - $\succ$ a (strict) preference ordering over P, and
  - Define $\succsim$ by p $\succsim$ q if q $\not\succ$ p

- *S* states of the world
- Set of all possible lotteries

$$P = \{p \in \mathbb{R}^S | p(c) \geq 0, \sum p(c) = 1\}$$

- Space with *S* dimensions

- Can we simplify the utility representation of preferences over lotteries?
- Space with *one* dimension – income
- We need to assume further axioms

# Expected Utility Theory

- A binary relation that satisfies the following three axioms if and only if there exists a function u(•) such that

$$p \succ q \Leftrightarrow \sum u(c)\, p(c) > \sum u(c)\, q(c)$$

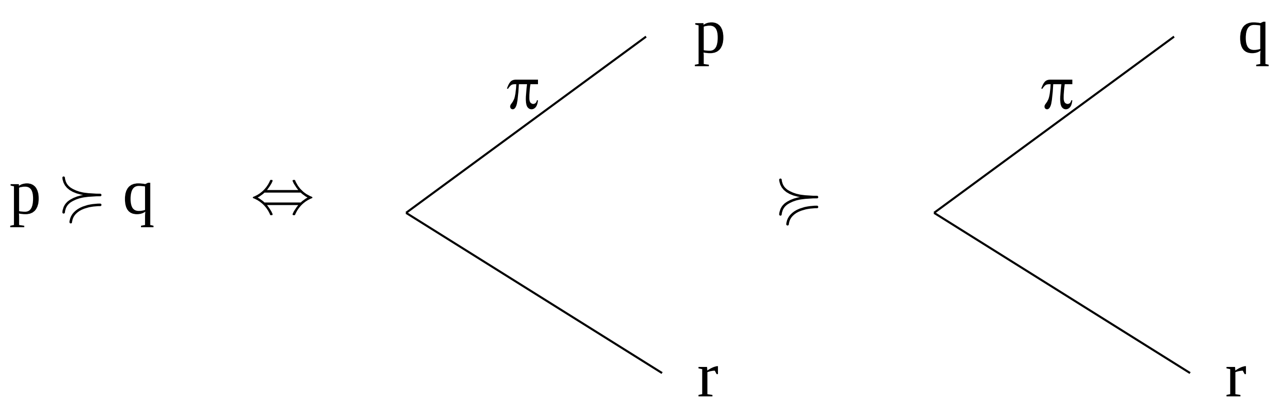
i.e. preferences correspond to expected utility.

# vNM Expected Utility Theory

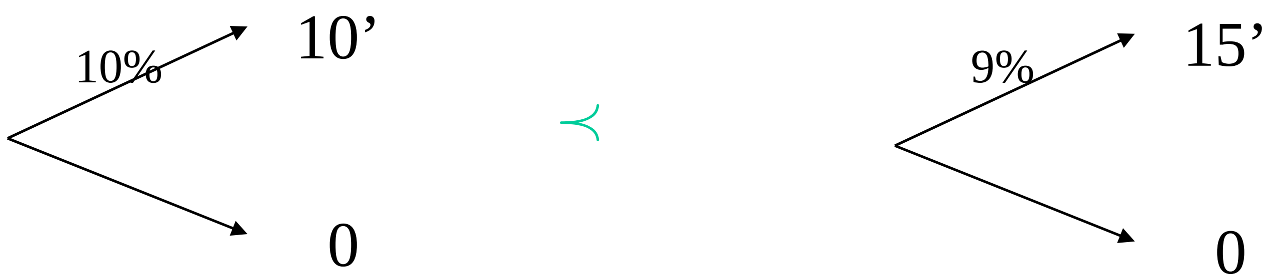
- Axiom 1 (Completeness and Transitivity):
  - Agents have preference relation over P (repeated)
- Axiom 2 (Substitution/Independence)
  - For all lotteries $p,q,r \in P$ and $\alpha \in (0,1]$,
    $p \succcurlyeq q$ iff $\alpha p + (1-\alpha) r \succcurlyeq \alpha q + (1-\alpha) r$ (see next slide)
- Axiom 3 (Archimedian/Continuity)
  - For all lotteries $p,q,r \in P$, if $p \succ q \succ r$, then there exists a $\alpha$ , $\beta \in (0,1)$ such that
    $\alpha p + (1- \alpha) r \succ q \succ \beta p + (1 - \beta) r$.
    *Problem:* p you get \$100 for sure, q you get \$ 10 for sure, r you are killed

# Independence Axiom

- Independence of irrelevant alternatives:



# Allais Paradox –
# Violation of Independence Axiom



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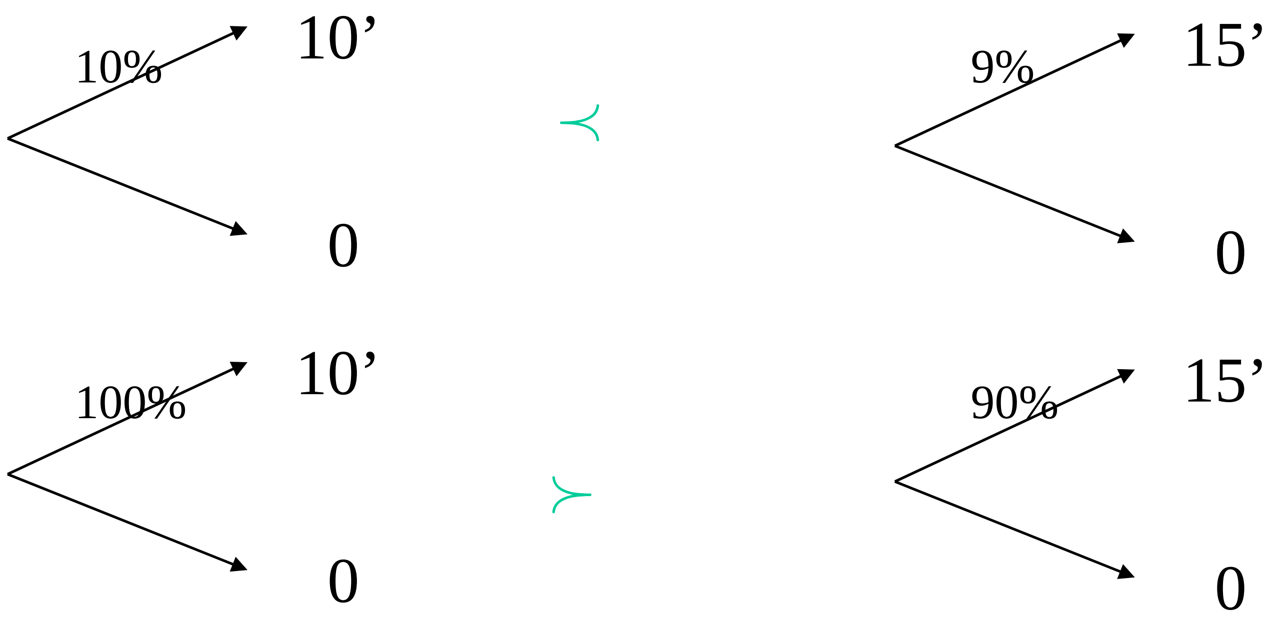
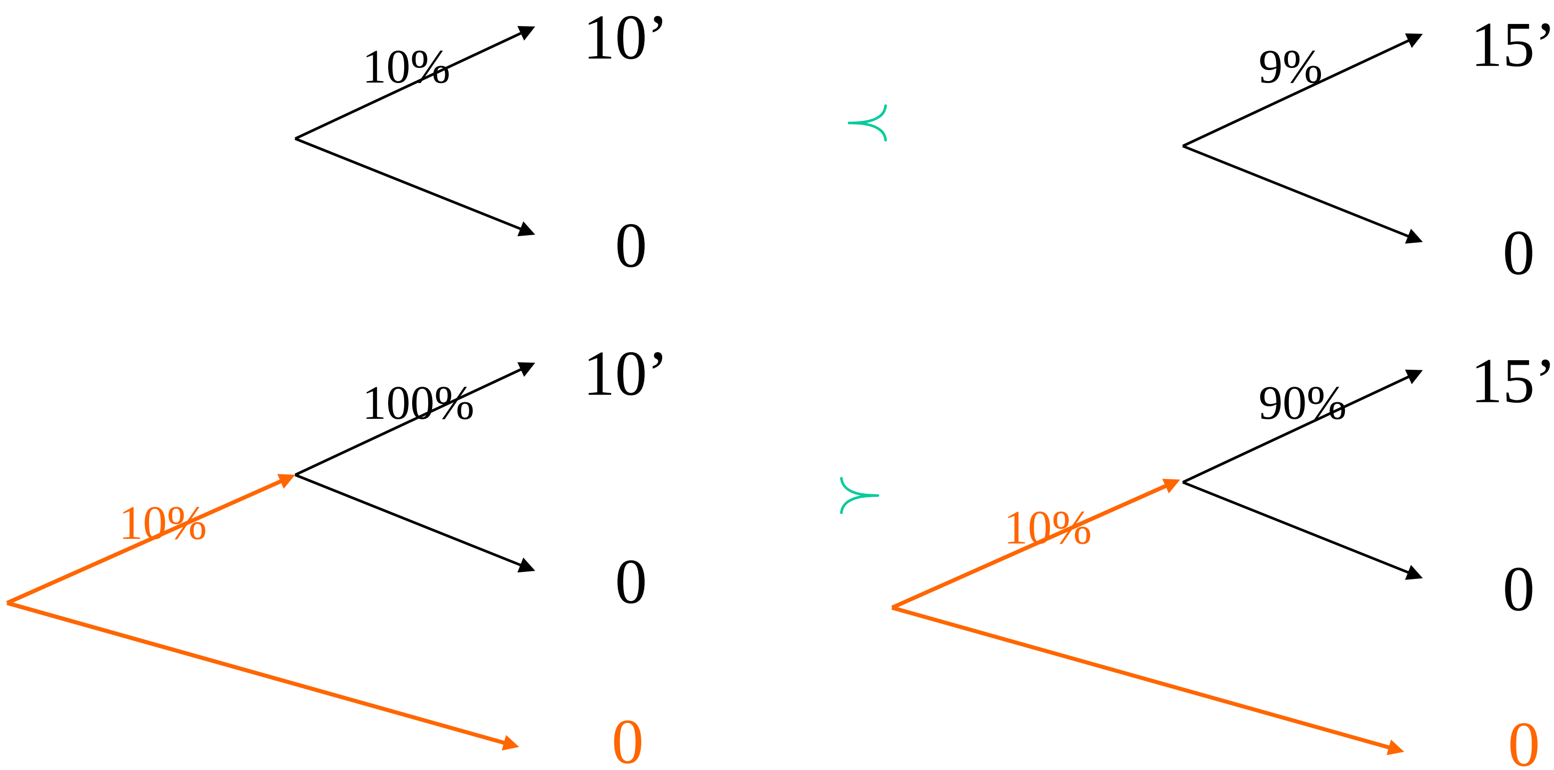


# Allais Paradox – Violation of Independence Axiom



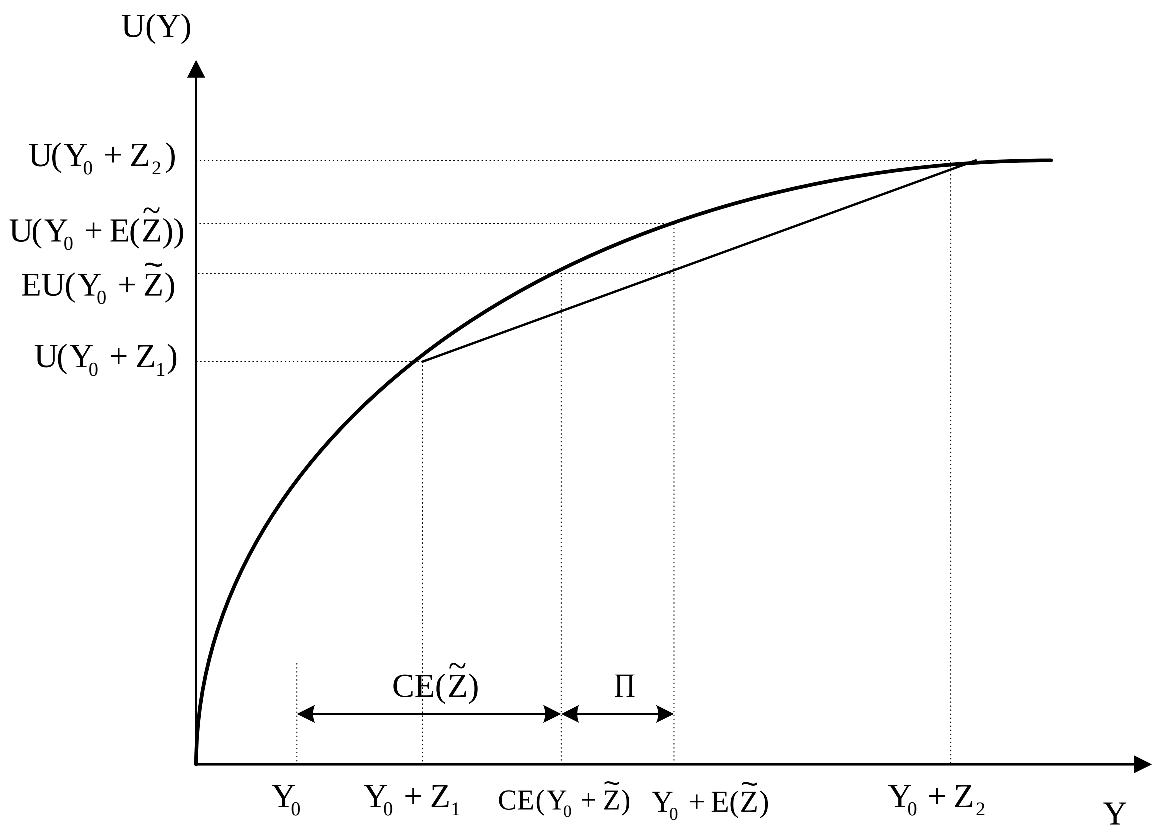
# vNM EU Theorem

- A binary relation that satisfies the axioms 1-3 if and only if there exists a function u(•) such that

$$p \succ q \Leftrightarrow \sum u(c)\, p(c) > \sum u(c)\, q(c)$$

i.e. preferences correspond to expected utility.

# Expected Utility Theory



# Expected Utility & Stochastic Dominance

- Theorem 3. 2 : *Let* $F_A(\tilde{x})$, $F_B(\tilde{x})$, *be two cumulative probability distribution for random payoffs* $\tilde{x} \in [a, b]$. *Then* $F_A(\tilde{x})$ FSD $F_B(\tilde{x})$ *if and only if for all non decreasing utility functions* $U(\bullet)$.

$$E_A U(\tilde{x}) \geq E_B U(\tilde{x})$$

# Expected Utility & Stochastic Dominance

- <u>Theorem 3. 3</u> : *Let* $F_A(\tilde{x})$ , $F_B(\tilde{x})$ , *be two cumulative probability distribution*
  *for random payoffs* $\tilde{x}$ *defined on* $[a, b]$ .
  Then, $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$ *if and only if* $E_A U(\tilde{x}) \geq E_B U(\tilde{x})$
  *for all non decreasing and concave* U.

# Digression: Subjective EU Theory

- Derive perceived probability from preferences!
  - Set S of prizes/consequences
  - Set Z of states
  - Set of functions $f(s) \in \mathcal{Z}$, called acts (consumption plans)
- Seven SAVAGE Axioms
  - Goes beyond scope of this course.

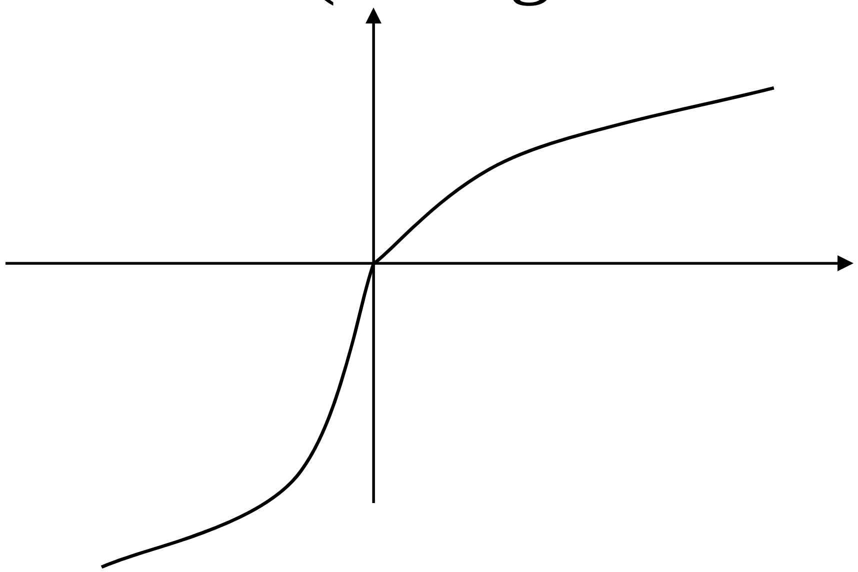
# Digression: Ellsberg Paradox

- 10 balls in an urn
  Lottery 1: win $100 if you draw a red ball
  Lottery 2: win $100 if you draw a blue ball
- Uncertainty: Probability distribution is not known
- Risk: Probability distribution is known
  (5 balls are red, 5 balls are blue)

- Individuals are *"uncertainty/ambiguity averse"*
  (non-additive probability approach)

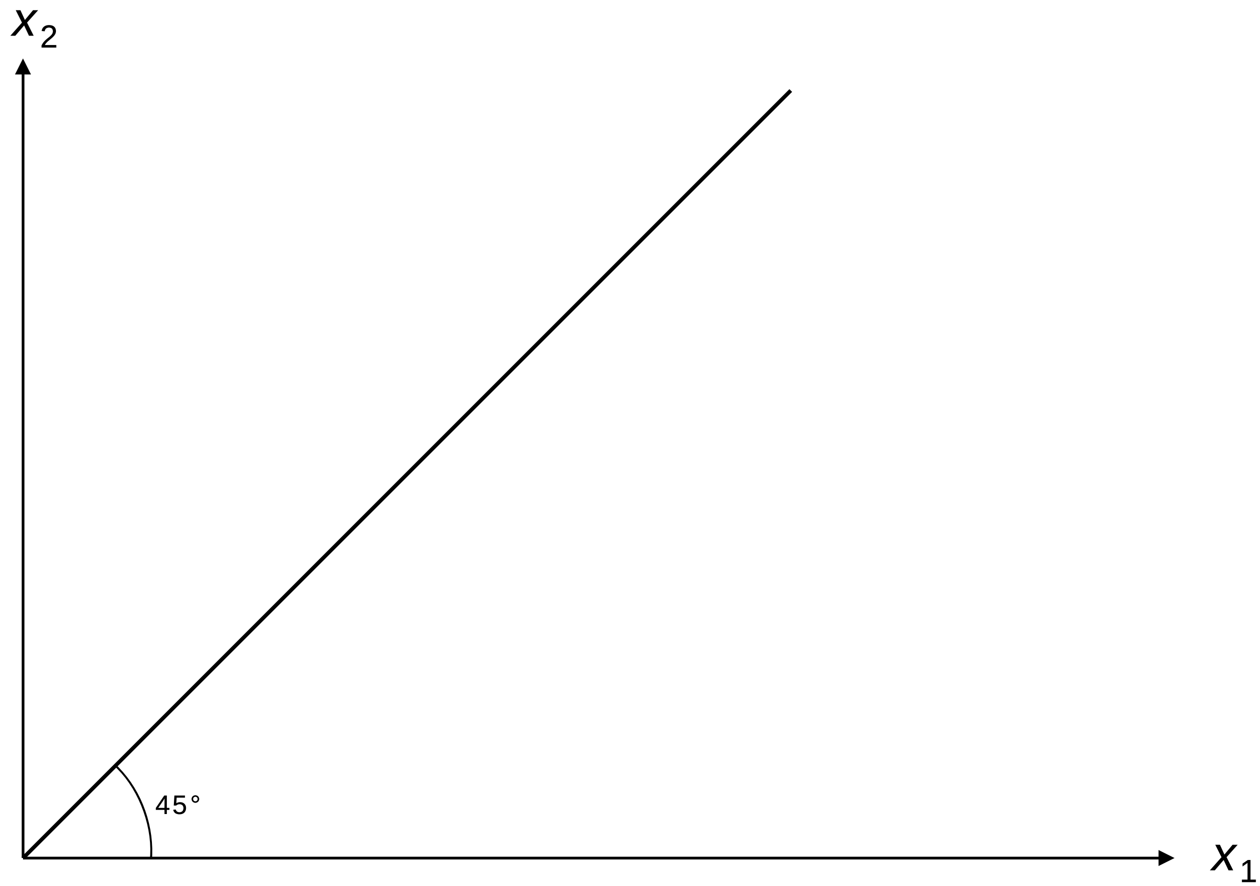
# Digression: Prospect Theory

- Value function (over gains and losses)



- Overweight low probability events
- Experimental evidence

# Indifference curves

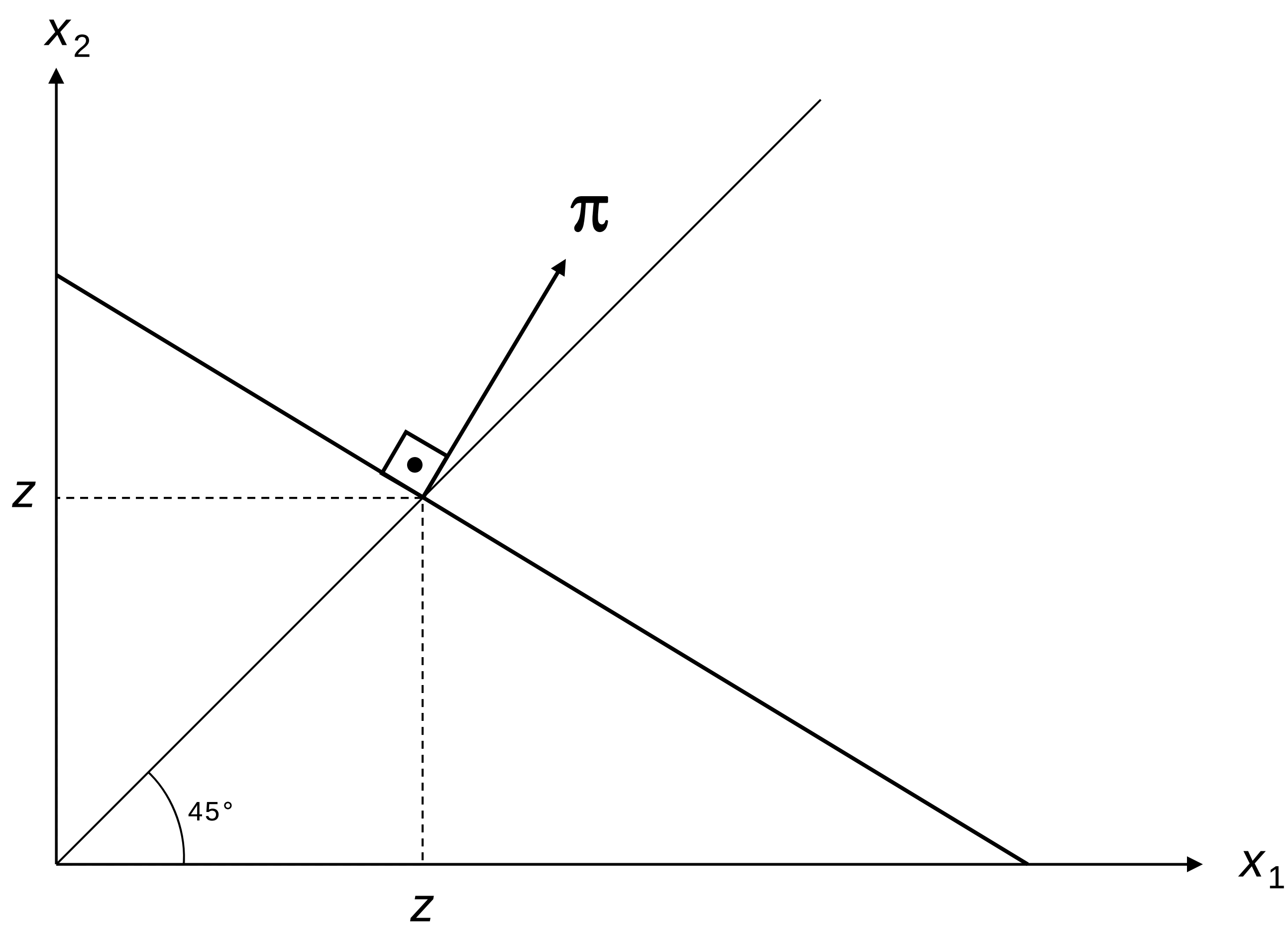



Any point in this plane is a particular lottery.

Where is the set of risk-free lotteries?

If $x_1 = x_2$, then the lottery contains no risk.

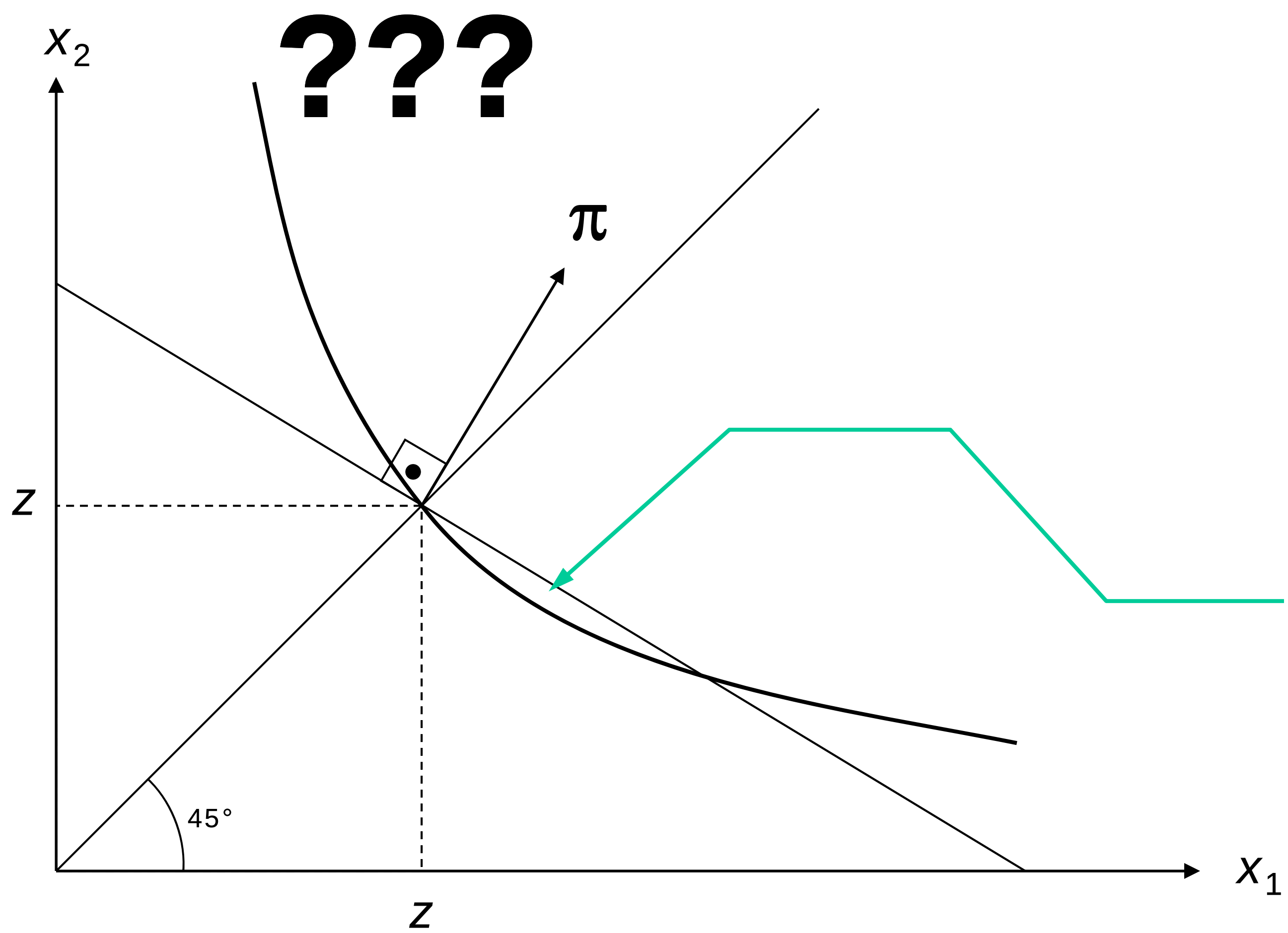
# Indifference curves




Where is the set of lotteries with expected prize $E[L]=z$?

It's a straight line, and the slope is given by the relative probabilities of the two states.

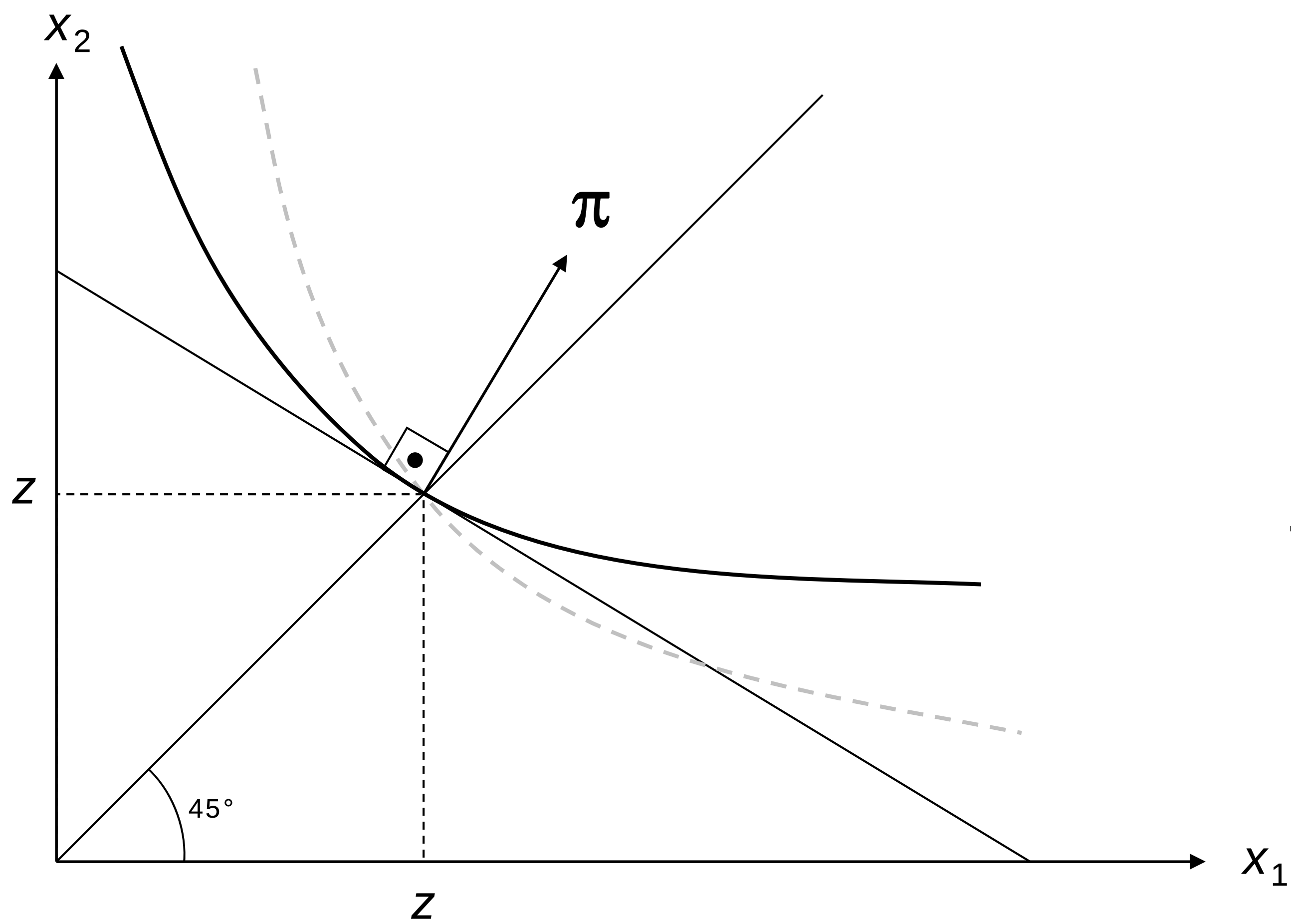
# Indifference curves




Suppose the agent is risk averse. Where is the set of lotteries which are indifferent to $(z,z)$?

That's not right! Note that there are risky lotteries with smaller expected prize and which are preferred.

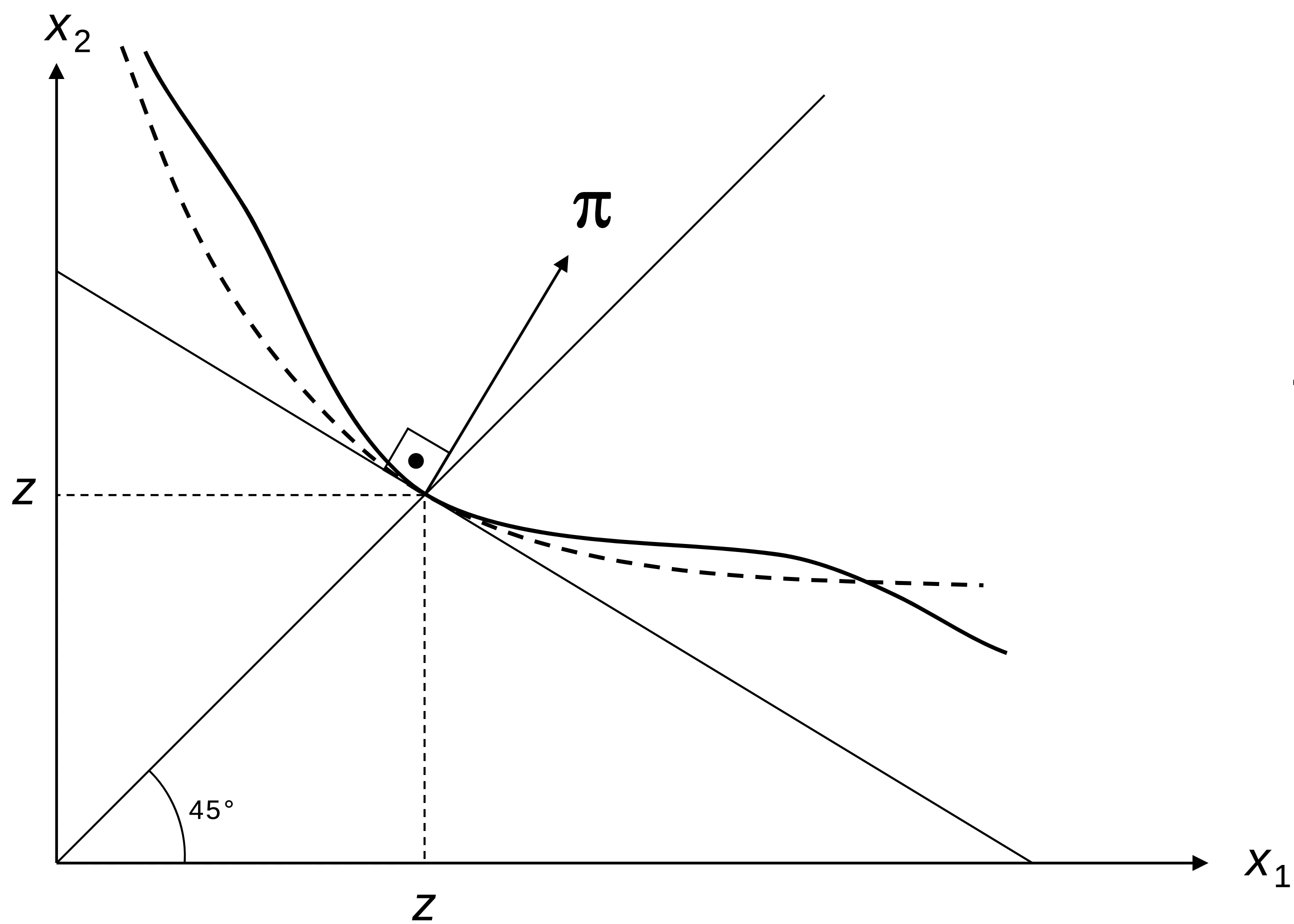
# Indifference curves




So the indifference curve must be tangent to the iso-expected-prize line.

This is a direct implication of risk-aversion alone.

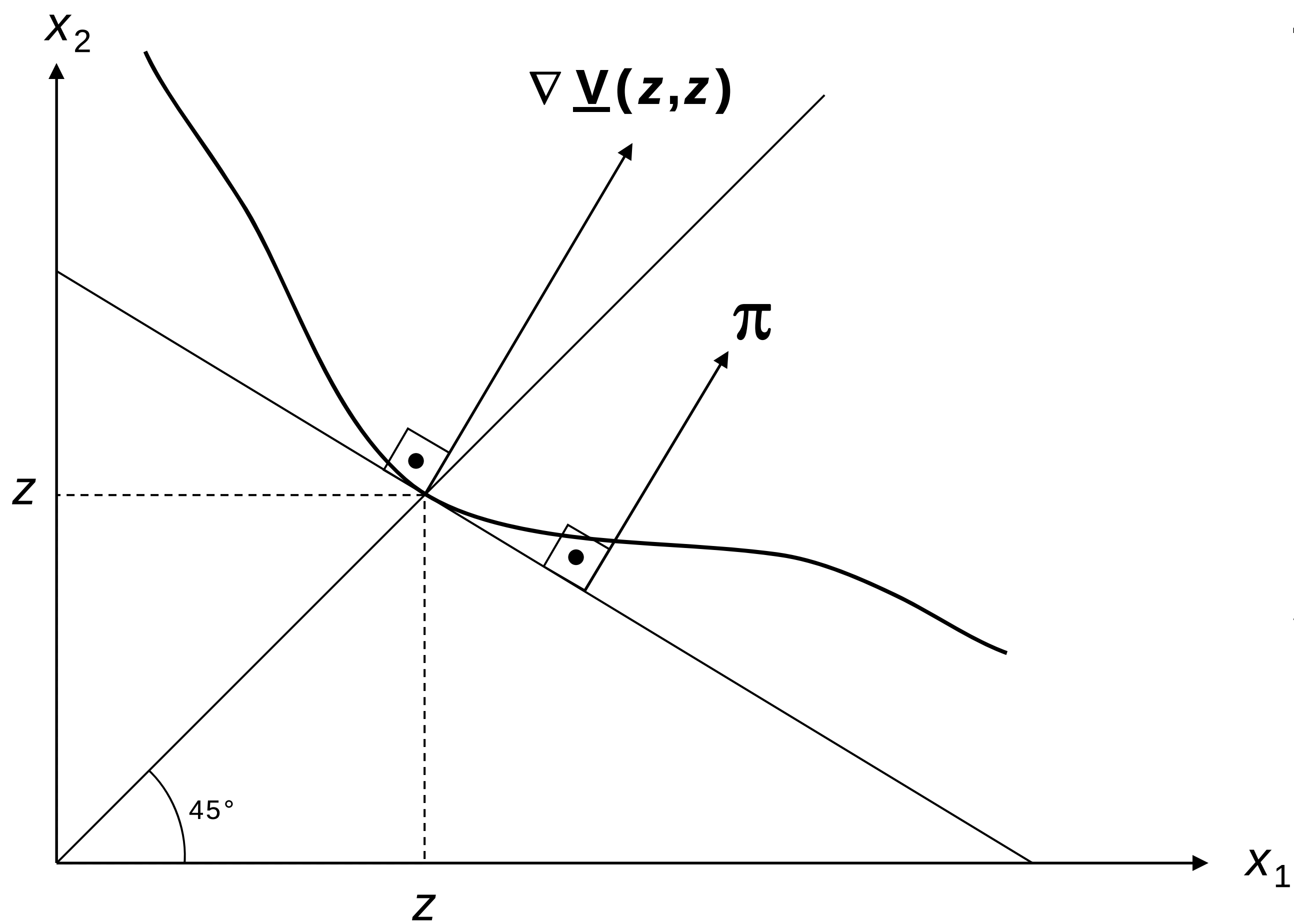
# Indifference curves




But risk-aversion does not imply convexity.

This indifference curve is also compatible with risk-aversion.

# Indifference curves




The tangency implies that the gradient of **V** at the point $(z,z)$ is collinear to $\pi$.

Formally, $\nabla$ **V**$(z,z) = \lambda\pi$, for some $\lambda>0$.

# Certainty Equivalent and Risk Premium

(3.6) $$EU(Y + \tilde{Z}) = U(Y + CE(Y, \tilde{Z}))$$

(3.7) $$= U(Y + E\tilde{Z} - \Pi(Y, \tilde{Z}))$$

# Certainty Equivalent and Risk Premium

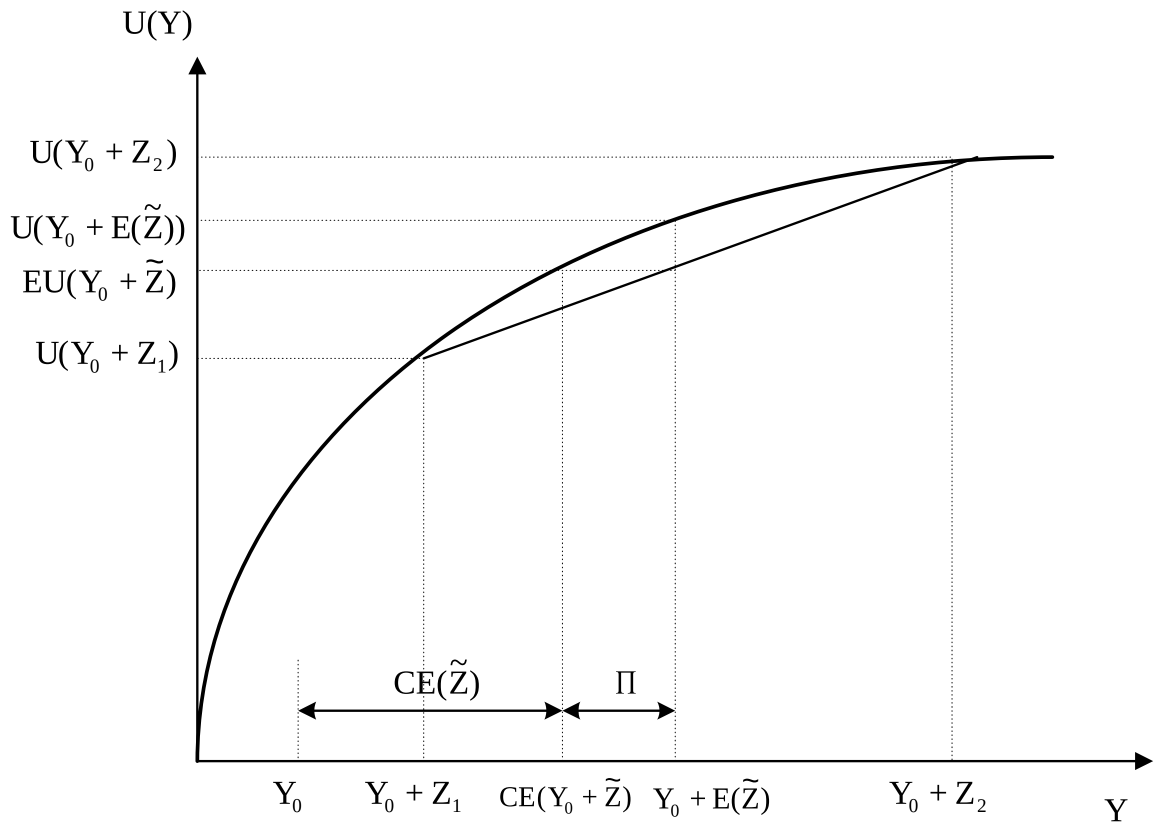



Figure 3-3 Certainty Equivalent and Risk Premium: An Illustration

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# Measuring Risk aversion

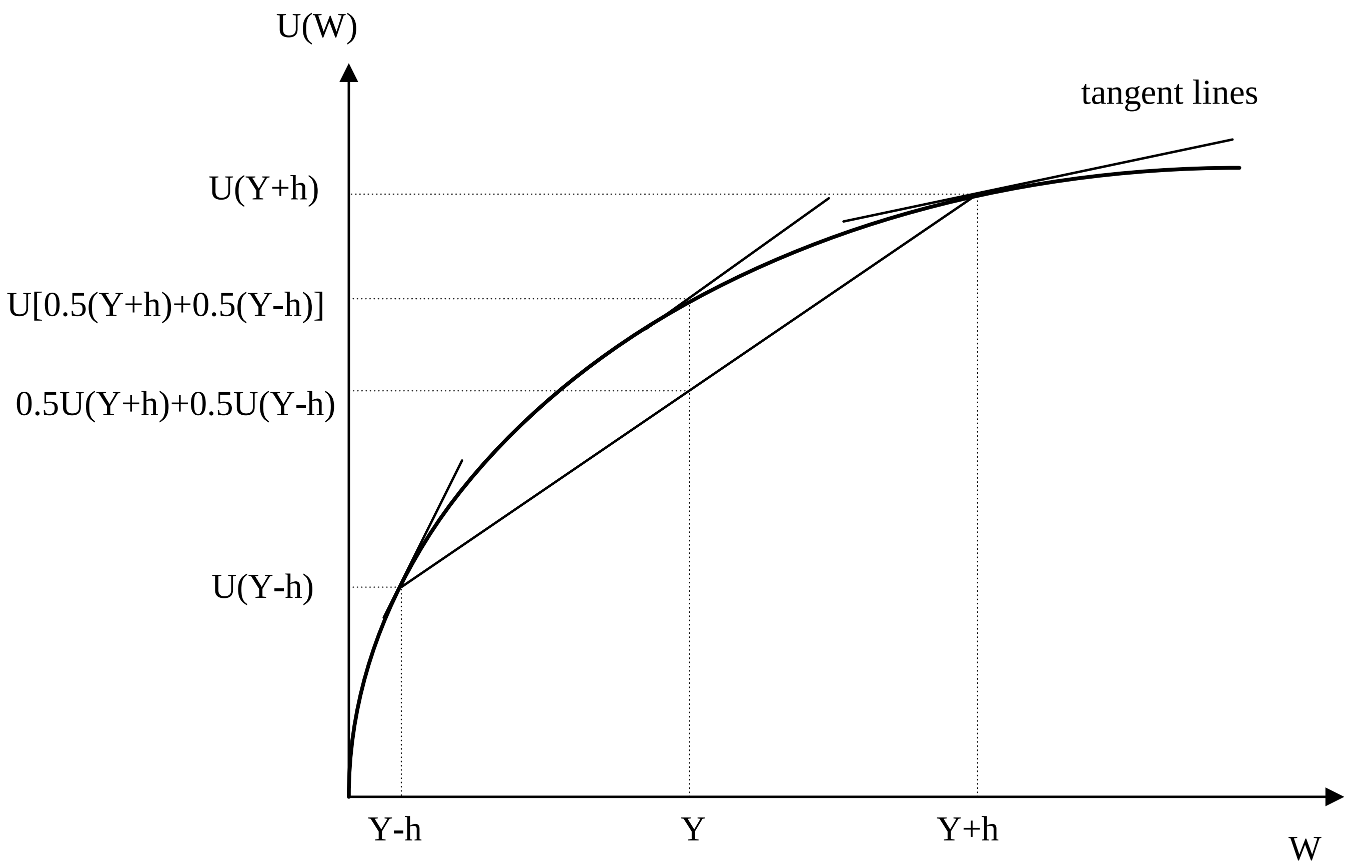


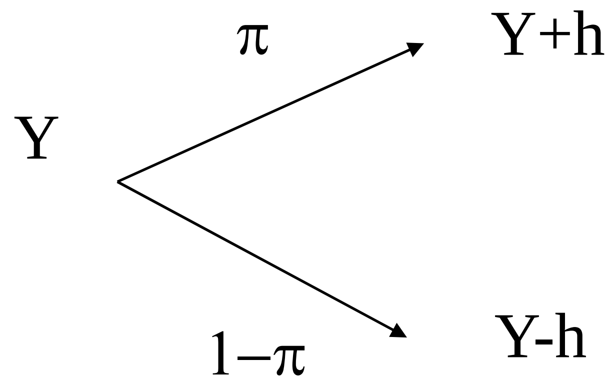

Figure 3-1 A Strictly Concave Utility Function

# Arrow-Pratt measures of risk aversion and their interpretations

- absolute risk aversion $= - \dfrac{U''(Y)}{U'(Y)} \equiv R_A(Y)$
- relative risk aversion $= - \dfrac{Y\,U''(Y)}{U'(Y)} \equiv R_R(Y)$
- risk tolerance $= \dfrac{1}{R_A}$

# Absolute risk aversion coefficient

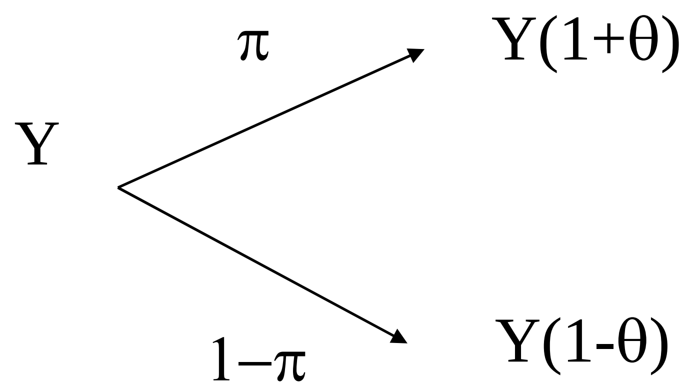
$$R_A = -\frac{U''(Y)}{U'(Y)}$$




$$\pi(Y,h) = \frac{1}{2} + \frac{1}{4} h R_A(Y) + HOT$$

# Relative risk aversion coefficient

$$R_R = -\frac{U''(Y)}{U'(Y)} Y$$




$$\pi(Y, \theta) = \frac{1}{2} + \frac{1}{4} \theta R_R(Y) + HOT$$

*Homework:* Derive this result.

# CARA and CRRA-utility functions

- Constant Absolute RA utility function

$$U(Y) = -e^{-\rho Y}$$

- Constant Relative RA utility function

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} \text{ for } \gamma \neq 1$$
$$U(Y) = lnY \quad \text{ for } \gamma = 1$$

# Investor ’s Level of Relative Risk Aversion

$$\frac{(Y+CE)^{1-\gamma}}{1-\gamma} = \frac{\frac{1}{2}(Y+50{,}000)^{1-\gamma}}{1-\gamma} + \frac{\frac{1}{2}(Y+100{,}000)^{1-\gamma}}{1-\gamma}$$

| | | |
|---|---|---|
| Y=0 | $\gamma = 0$ | CE = 75,000 (risk neutrality) |
| | $\gamma = 1$ | CE = 70,711 |
| | $\gamma = 2$ | CE = 66,246 |
| | $\gamma = 5$ | CE = 58,566 |
| | $\gamma = 10$ | CE = 53,991 |
| | $\gamma = 20$ | CE = 51,858 |
| | $\gamma = 30$ | CE = 51,209 |
| Y=100,000 | $\gamma = 5$ | CE = 66,530 |

# Risk aversion and Portfolio Allocation

- No savings decision (consumption occurs only at $t=1$)
- Asset structure
  - One risk free bond with net return $r_f$
  - One risky asset with random net return $r$ (a =quantity of risky assets)

$$\max_a E[U(Y_0(1+r_f)+a(r-r_f))]$$

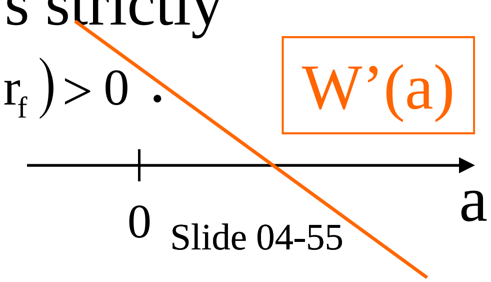
FOC:

$$E[U'(Y_0(1+r_f)+a(r-r_f))(r-r_f)]=0$$

- Theorem 4.1: Assume $U'(\ ) > 0$, and $U''(\ ) < 0$ and let $\hat{a}$ denote the solution to above problem. Then

$$\hat{a} > 0 \quad \text{if and only if } E\tilde{r} > r_f$$
$$\hat{a} = 0 \quad \text{if and only if } E\tilde{r} = r_f$$
$$\hat{a} < 0 \quad \text{if and only if } E\tilde{r} < r_f \ .$$

- Define $W(a) = E\{U(Y_0(1+r_f) + a(\tilde{r} - r_f))\}$. The FOC can then be written $W'(a) = E[U'(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)] = 0$ . By risk aversion ($U''<0$), $W''(a) = E\left[U''(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)^2\right]$ $< 0$, that is, $W'(a)$ is everywhere decreasing. It follows that $\hat{a}$ will be positive if and only if $W'(0) = U'(Y_0(1+r_f))E(\tilde{r} - r_f) > 0$ (since then a will have to be increased from its value of 0 to achieve equality in the FOC). Since $U'$ is always strictly positive, this implies $\hat{a} > 0$ if and only if $E(\tilde{r} - r_f) > 0$ .
The other assertion follows similarly. □



# Portfolio as wealth changes

- Theorem 4.4 (Arrow, 1971): Let $\hat{a} = \hat{a}(Y_0)$ be the solution to max-problem above; then:

(i) $\frac{\partial R_A}{\partial Y} < 0$ (DARA) implies $\frac{\partial \hat{a}}{\partial Y_0} > 0$

(ii) $\frac{\partial R_A}{\partial Y} = 0$ (CARA) implies $\frac{\partial \hat{a}}{\partial Y_0} = 0$

(iii) $\frac{\partial R_A}{\partial Y} > 0$ (IARA) implies $\frac{\partial \hat{a}}{\partial Y_0} < 0$

# Portfolio as wealth changes

- <u>Theorem 4.5 (Arrow 1971):</u> If, for all wealth levels Y,

(i) $\frac{\partial R_R}{\partial Y} = 0 \text{ (CRRA) implies } \eta = 1$

(ii) $\frac{\partial R_R}{\partial Y} < 0 \text{ (DRRA) implies } \eta > 1$

(iii) $\frac{\partial R_R}{\partial Y} > 0 \text{ (IRRA) implies } \eta < 1$

where $\eta = \frac{da/a}{dY/Y}$ (elasticity)

# Log utility & Portfolio Allocation

U(Y) = ln Y.

$$E\left\{\frac{\tilde{r}-r_f}{Y_0(1+r_f)+a(\tilde{r}-r_f)}\right\}=0$$

2 states, where $r_2 > r_f > r_1$

$$\frac{a}{Y_0}=\frac{(1+r_f)[E[\tilde{r}]-r_f]}{-(r_1-r_f)(r_2-r_f)}>0$$

Constant fraction of wealth is invested in risky asset!

# Portfolio of risky assets as wealth changes

## Now -- many risky assets

- <u>Theorem 4.6 (Cass and Stiglitz,1970).</u> Let the vector

$$\begin{bmatrix} \hat{a}_1(Y_0) \\ \cdot \\ \cdot \\ \hat{a}_J(Y_0) \end{bmatrix}$$ denote the amount optimally invested in the *J* risky assets if

the wealth level is $Y_0$. Then
$$\begin{bmatrix} \hat{a}_1(Y_0) \\ \cdot \\ \cdot \\ \hat{a}_J(Y_0) \end{bmatrix} = \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ a_J \end{bmatrix} f(Y_0)$$

if and only if either

(i) $U'(Y_0) = (\theta Y_0 + \kappa)^{\Delta}$ or

(ii) $U'(Y_0) = \xi e^{-\nu Y_0}$ .

- In words, it is sufficient to offer a **mutual fund.**

# LRT/HARA-utility functions

- Linear Risk Tolerance/hyperbolic absolute risk aversion

$$-\frac{u''(c)}{u'(c)} = \frac{1}{A+Bc}$$

- Special Cases
  - B=0, A>0 CARA
  - $B \neq 0, \neq 1$ Generalized Power $u(c) = \frac{1}{B-1}(A+Bc)^{\frac{B-1}{B}}$
    - B=1 Log utility u(c) =ln (A+Bc)
    - B=-1 Quadratic Utility $u(c)=-(A-c)^2$
    - $B \neq 1$ A=0 CRRA Utility function $u(c) = \frac{1}{B-1}(Bc)^{\frac{B-1}{B}}$

# Overview: Risk Preferences

1. State-by-state dominance
2. Stochastic dominance [DD4]
3. vNM expected utility theory
   a) Intuition [L4]
   b) Axiomatic foundations [DD3]
4. Risk aversion coefficients and portfolio choice [DD4,5,L4]
5. Prudence coefficient and precautionary savings [DD5]
6. Mean-variance preferences [L4.6]

# Introducing Savings

- Introduce savings decision: Consumption at $t=0$ and $t=1$
- *Asset structure 1:*
  - risk free bond $R^f$
  - NO risky asset with random return

– Increase $R^f$:
  - **Substitution effect:** shift consumption from $t$=0 to $t$=1
    $\Rightarrow$ save more
  - **Income effect:** agent is "effectively richer" and wants to consume some of the additional richness at $t$=0
    $\Rightarrow$ save less
  - For log-utility ($\gamma$=1) both effects cancel each other

# Prudence and Pre-cautionary Savings

- Introduce savings decision
  Consumption at $t=0$ and $t=1$
- *Asset structure 2:*
  - NO risk free bond
  - One risky asset with random gross return $R$

# Prudence and Savings Behavior

- Risk aversion is about the willingness to insure …
- … but not about its comparative statics.
- How does the behavior of an agent change when we marginally increase his exposure to risk?
- An old hypothesis (going back at least to J.M.Keynes) is that people should save more now when they face greater uncertainty in the future.
- The idea is called *precautionary saving* and has intuitive appeal.

# Prudence and Pre-cautionary Savings

- Does not directly follow from risk aversion alone.
- Involves the third derivative of the utility function.
- Kimball (1990) defines **absolute prudence** as

$$P(w) := -u'''(w)/u''(w).$$

- Precautionary saving if any only if they are prudent.
- This finding is important when one does comparative statics of interest rates.
- Prudence seems uncontroversial, because it is weaker than DARA.

# Pre-cautionary Saving (extra material)

$$\max_s E[U(Y_0 - s) + \delta U(sR)]$$

$$\text{s.t. } s \geq 0$$

$$\text{FOC: } U'(Y_0 - s) = \delta E[U'(sR)R]$$

(+) (-) in s

Is saving *s* increasing/decreasing in risk of *R*?
Is RHS increasing/decreasing is riskiness of *R*?
Is *U'*() convex/concave?
Depends on third derivative of *U*()!

N.B: For $U(c)=\ln c$, $U'(sR)R=1/s$ does not depend on R.

# Pre-cautionary Saving (extra material)

**2 effects:** Tomorrow consumption is more volatile

- consume more today, since it's not risky
- save more for precautionary reasons

Theorem 4.7 (Rothschild and Stiglitz,1971) : Let $\tilde{R}_A$ , $\tilde{R}_B$ be two return distributions with identical means such that $\tilde{R}_B = \tilde{R}_A + e$, (where *e* is white noise) and let $s_A$ and $s_B$ be, respectively, the savings out of $Y_0$ corresponding to the return distributions $\tilde{R}_A$ and $\tilde{R}_B$.

If $R'_R(Y) \leq 0$ and $R_R(Y) > 1$, then $s_A < s_B$ ;

If $R'_R(Y) \geq 0$ and $R_R(Y) < 1$, then $s_A > s_B$

# Prudence & Pre-cautionary Saving

$$\mathbf{P}(c) = \frac{-U'''(c)}{U''(c)}$$

$$\mathbf{P}(c)c = \frac{-cU'''(c)}{U''(c)}$$

- <u>Theorem 4.8</u> : Let $\tilde{R}_A, \tilde{R}_B$ be two return distributions such that $\tilde{R}_A$ SSD $\tilde{R}_B$, and let $s_A$ and $s_B$ be, respectively, the savings out of $Y_0$ corresponding to the return distributions $\tilde{R}_A$ and $\tilde{R}_B$. Then,

$$s_A \geq s_B \quad \text{iff} \quad c\mathbf{P}(c) \leq 2, \text{ and conversely,}$$

$$s_A < s_B \quad \text{iff} \quad c\mathbf{P}(c) > 2$$

# Overview: Risk Preferences

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6. Mean-variance preferences [L4.6]

# Mean-variance Preferences

- Early researchers in finance, such as Markowitz and Sharpe, used just the mean and the variance of the return rate of an asset to describe it.
- Mean-variance characterization is often easier than using an vNM utility function
- But is it compatible with vNM theory?
- The answer is yes … approximately … under some conditions.

# Mean-Variance: quadratic utility

Suppose utility is quadratic, $u(y) = ay - by^2$.

Expected utility is then

$$E[u(y)] = aE[y] - bE[y^2]$$

$$= aE[y] - b\left(E[y]^2 + \mathrm{var}[y]\right).$$

Thus, expected utility is a function of the mean, $E[y]$, and the variance, $\mathrm{var}[y]$, only.

# Mean-Variance: joint normals

- Suppose all lotteries in the domain have normally distributed prized. (independence is not needed).
  - This requires an infinite state space.
- Any linear combination of normals is also normal.
- The normal distribution is completely described by its first two moments.
- Hence, expected utility can be expressed as a function of just these two numbers as well.

# Mean-Variance: linear distribution classes

- Generalization of joint nomarls.
- Consider a class of distributions $F_1, \ldots, F_n$ with the following property:
  - for all $i$ there exists $(m,s)$ such that $F_i(x) = F_1(a+bx)$ for all $x$.
- This is called a linear distribution class.
- It means that any $F_i$ can be transformed into an $F_j$ by an appropriate shift ($a$) and stretch ($b$).
- Let $y_i$ be a random variable drawn from $F_i$. Let $\mu_i = E\{y_i\}$ and $\sigma_i^2 = E\{(y_i - \mu_i)^2\}$ denote the mean and the var of $y_i$.

# Mean-Variance: linear distribution classes

- Define then the random variable $x = (y_i - \mu_i)/\sigma_i$. We denote the distribution of $x$ with $F$.
- Note that the mean of $x$ is 0 and the variance is 1, and F is part of the same linear distribution class.
- Moreover, the distribution of $x$ is independent of which $i$ we start with.
  - We want to evaluate the expected utility of $y_i$,

$$\int_{-\infty}^{+\infty} v(z) dF_i(z).$$

# Mean-Variance: linear distribution classes

But $y_i = \mu_i + \sigma_i x$, thus

$$\int_{-\infty}^{+\infty} v(z)dF_i(z) = \int_{-\infty}^{+\infty} v(\mu_i + \sigma_i z)dF(z)$$

$$=: u(\mu_i, \sigma_i).$$

The expected utility of all random variables drawn from the same linear distribution class can be expressed as functions of the mean and the standard deviation only.

# Mean-Variance: small risks

- Justification for mean-variance for the case of small risks.
- use a second order (local) Taylor approximation of vNM U(c).
- If U(c) is concave, second order Taylor approximation is a quadratic function with a negative coefficient on the quadratic term.
- Expectation of a quadratic utility function can be evaluated with the mean and variance.

# Mean-Variance: small risks

- Let $f$ : R → R be a smooth function. The Taylor approximation is

$$f(x) \approx f(x_0) + f'(x_0)\frac{(x-x_0)^1}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

- Use Taylor approximation for E[u(x)].

# Mean-Variance: small risks

- Since E[u($w$+$x$)] = u($c^{CE}$), this simplifies to

$$w - c_{CE} \approx R_A(w)\frac{\mathrm{var}(x)}{2}.$$

  - $w$ – $c^{CE}$ is the risk premium.
  - We see here that the risk premium is approximately a linear function of the variance of the additive risk, with the slope of the effect equal to half the coefficient of absolute risk.

# Mean-Variance: small risks

- The same exercise can be done with a multiplicative risk.
- Let $y = gw$, where $g$ is a positive random variable with unit mean.
- Doing the same steps as before leads to

$$1-\kappa \approx R_R(w)\frac{\mathrm{var}(g)}{2},$$

where $\kappa$ is the certainty equivalent growth rate, $u(\kappa w) = E[u(gw)]$.

- The coefficient of *relative* risk aversion is relevant for multiplicative risk, absolute risk aversion for additive risk.

# Extra material follows!

# Joint saving-portfolio problem

- Consumption at $t=0$ and $t=1$. (savings decision)
- Asset structure
  - One risk free bond with net return $r_f$
  - One risky asset (a = quantity of risky assets)

$$\max_{\{a,s\}} U(Y_0 - s) + \delta E U(s(1+r_f) + a(\tilde{r} - r_f)) \qquad (4.7)$$

*FOC:*

s: $U'(c_t) = \delta E[U'(c_{t+1})(1+r_f)]$

a: $E[U'(c_{t+1})(r-r_f)] = 0$

*for CRRA utility functions*

$$s: \quad (Y_0 - s)^{-\gamma}(-1) + \delta E\Big([s(1+r_f) + a(\tilde{r} - r_f)]^{-\gamma}(1+r_f)\Big) = 0$$

$$a: \quad E\Big[(s(1+r_f) + a(\tilde{r} - r_f))^{-\gamma}(\tilde{r} - r_f)\Big] = 0$$

Where *s* is total saving and *a* is amount invested in risky asset.

# Multi-period Setting

- Canonical framework (exponential discounting)

  $$U(c) = E[\sum \beta^t u(c_t)]$$

  - prefers earlier uncertainty resolution if it affect action
  - indifferent, if it does not affect action
- Time-inconsistent (hyperbolic discounting)

  Special case: $\beta$–$\delta$ formulation

  $$U(c) = E[u(c_0) + \beta \sum \delta^t u(c_t)]$$

- Preference for the timing of uncertainty resolution

  recursive utility formulation (Kreps-Porteus 1978)

# Multi-period Portfolio Choice

$$\max_{\{s_t,a_t\}_{t=0}^{T-1}} E[\textstyle\sum_{t=0}^{T} \beta^t U(c_t)]$$

s.t.

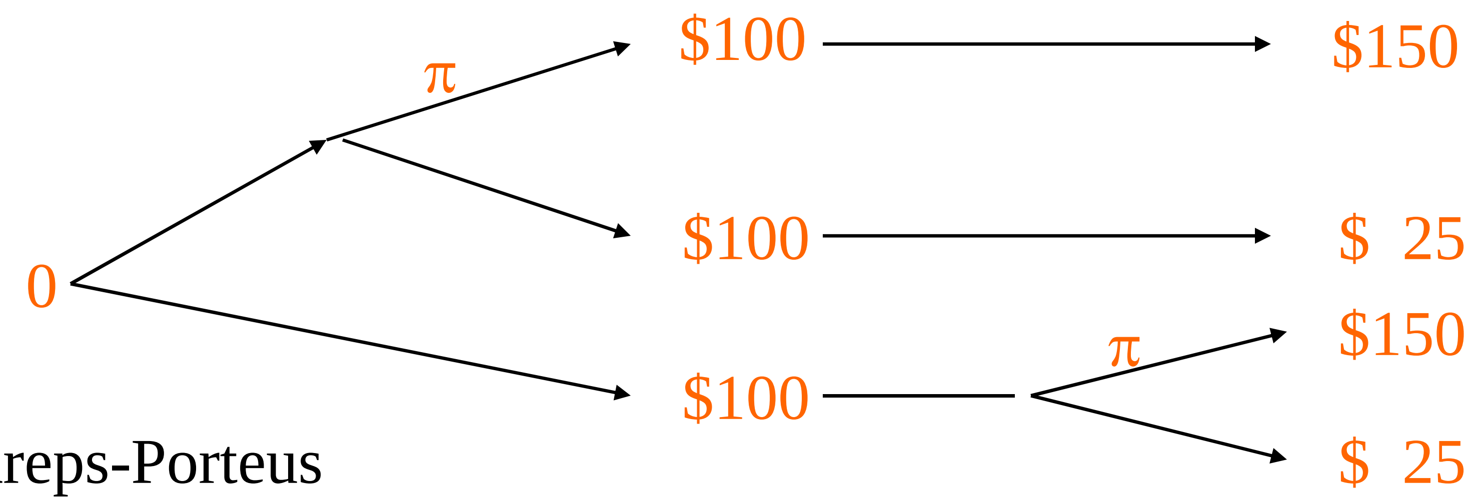
$$c_T = s_{T-1}(1+r_f) + a_{T-1}(r_T - r_f)$$

$$c_t + s_t \leq s_{t-1}(1+r_f) + a_{t-1}(r_t - r_f)$$

$$c_0 + s_0 \leq Y_0$$

<u>Theorem 4.10 (Merton, 1971):</u> Consider the above canonical multi-period consumption-saving-portfolio allocation problem. Suppose U() displays CRRA, $r_f$ is constant and {r} is i.i.d. Then $a/s_t$ is time invariant.

# Digression: Preference for the timing of uncertainty resolution




Kreps-Porteus

$$U_0(x_1, x_2(s)) = W(x_1, E[U_1(x_1, x_2(s))])$$

Early (late) resolution if $W(P_1,\ldots)$ is convex (concave)

Marginal rate of temporal substitution 🕒 risk aversion