Lecture 10: Multi-period Model Options – Black-Scholes-Merton model

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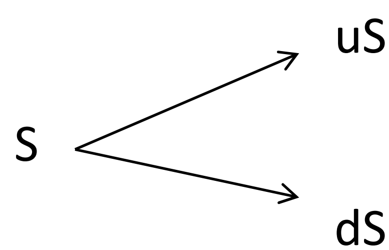
Binomial Option Pricing

- Consider a European call option maturing at time T with strike K: $C_T=\max(S_T-K,0)$, no cash flows in between
- Not able to statically replicate this payoff using just the stock and risk-free bond
- Need to *dynamically hedge* – required stock position changes for each period until maturity
  - static hedge for forward, using put-call parity
- Replication strategy depends on specified random process of stock price – need to know how price evolves over time. Binomial (Cox-Ross-Rubinstein) model is canonical

Assumptions

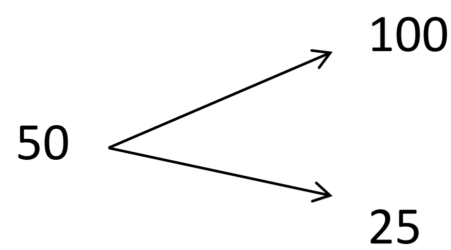
- Assumptions:
  - Stock which pays no dividend
  - Over each period of time, stock price moves from S to either uS or dS, i.i.d. over time, so that final distribution of S_T is binomial



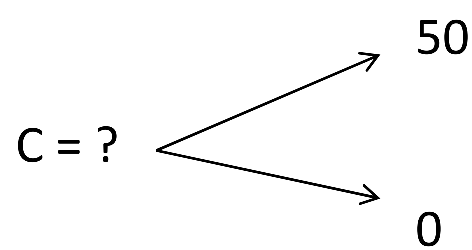
  - Suppose length of period is h and risk-free rate is given by $R = e^{rh}$
  - No arbitrage: $u > R > d$
  - Note: simplistic model, but as we will see, with enough periods begins to look more realistic

A One-Period Binomial Tree

- Example of a single-period model
  - S=50, u = 2, d= 0.5, R=1.25



  - What is value of a European call option with K=50?
  - Option payoff: $\max(S_T\text{-}K,0)$



  - Use replication to price

Single-period replication

- Consider a long position of Δ in the stock and B dollars in bond
- Payoff from portfolio:

$\Delta S+B=50\Delta+B$ → $\Delta uS+RB=100\Delta+1.25B$

$\Delta S+B=50\Delta+B$ → $\Delta dS+RB=25\Delta+1.25B$

- Define C_u as option payoff in up state and C_d as option payoff in down state ($C_u=50, C_d=0$ here)
- Replicating strategy must match payoffs:

$$C_u=\Delta uS+RB$$

$$C_d=\Delta dS+RB$$

Single-period replication

- Solving these equations yields:

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = \frac{uC_d - dC_u}{R(u-d)}$$

- In previous example, Δ=2/3 and B=-13.33, so the option value is

$$C = \Delta S + B = 20$$

- Interpretation of Δ: sensitivity of call price to a change in the stock price. Equivalently, tells you how to hedge risk of option
  - To hedge a long position in call, short Δ shares of stock

Risk-neutral probabilities

- Substituting Δ and B from into formula for C,

$$C = \frac{C_u - C_d}{S(u-d)} S + \frac{uC_d - dC_u}{R(u-d)}$$

$$= \frac{1}{R}\left[\frac{R-d}{u-d} C_u + \frac{u-R}{u-d} C_d\right]$$

- Define p = (R-d)/(u-d), note that 1-p = (u-R)/(u-d), so

$$C = \frac{1}{R}\left[pC_u + (1-p)C_d\right]$$

- Interpretation of p: probability the stock goes to uS in world where everyone is risk-neutral

Risk-neutral probabilities

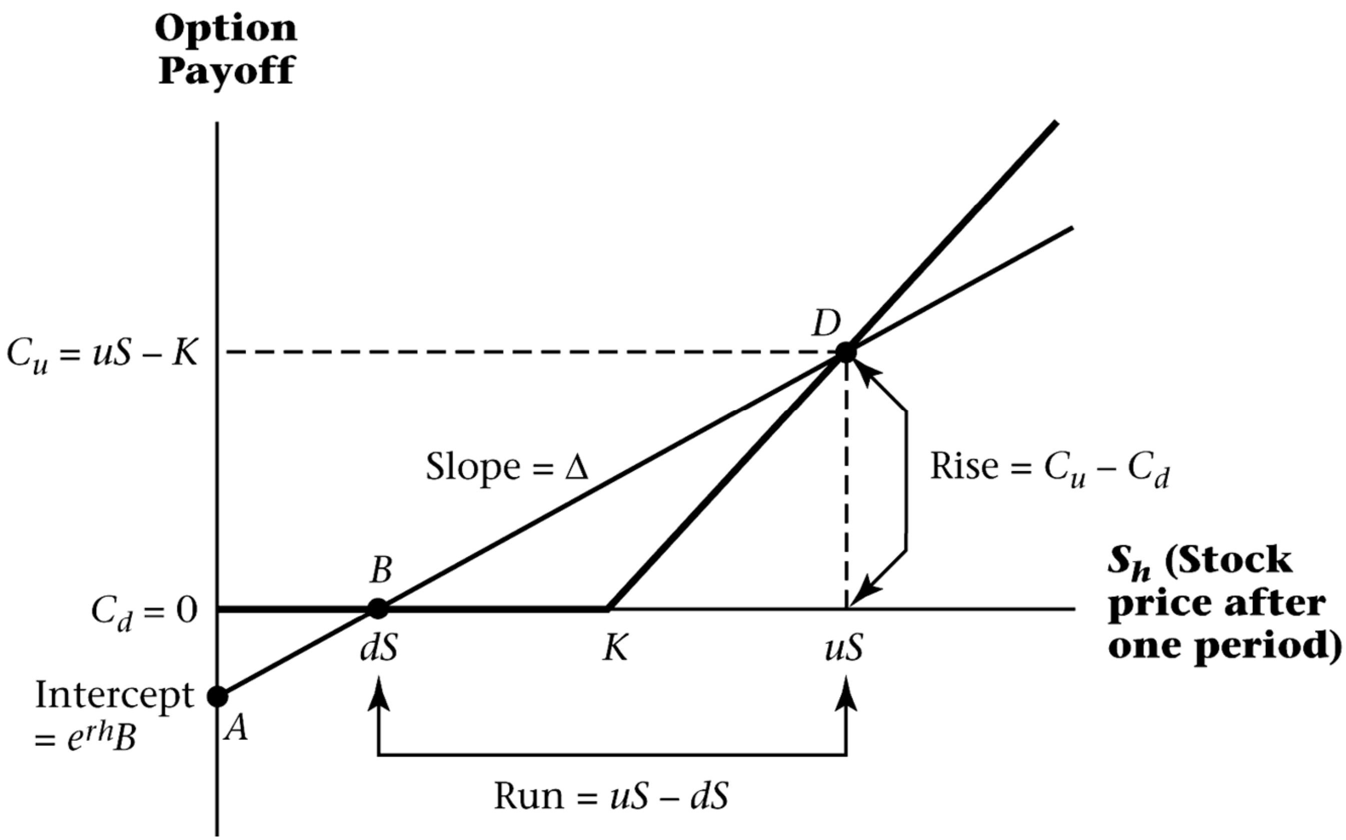
- Note that p is the probability that would justify the current stock price S in a risk-neutral world:

$$S = \frac{1}{R}\left[quS + (1-q)dS\right]$$

$$q = \frac{R-d}{u-d} = p$$

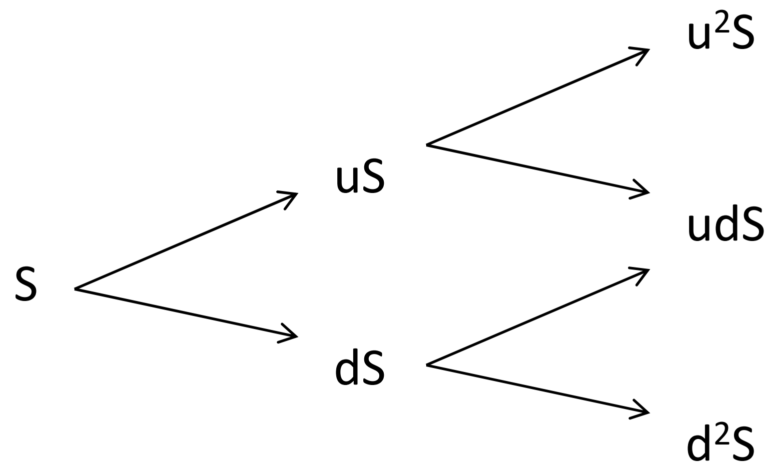
- No arbitrage requires u > R > d as claimed before
- Note: didn’t need to know anything about the objective probability of stock going up or down (P-measure). Just need a model of stock prices to construct Q-measure and price the option.

The binomial formula in a graph



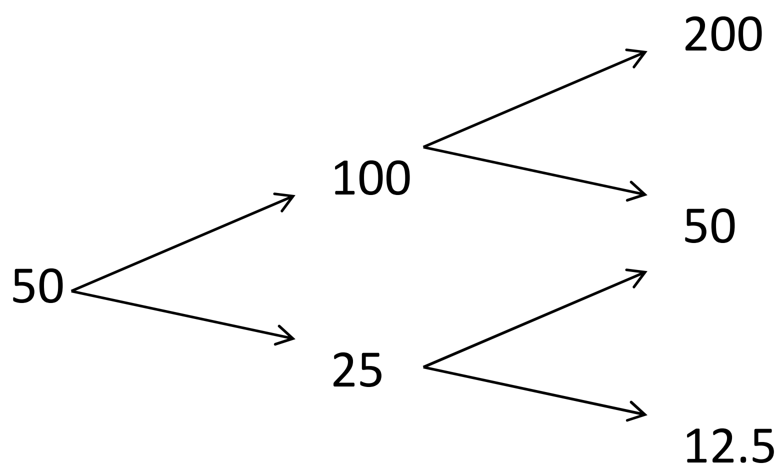
Two-period binomial tree

- Concatenation of single-period trees:

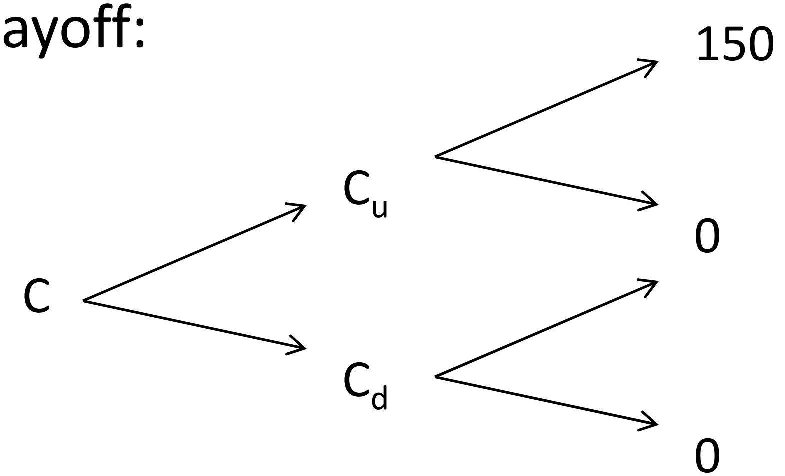


Two-period binomial tree

- Example: S=50, u=2, d=0.5, R=1.25

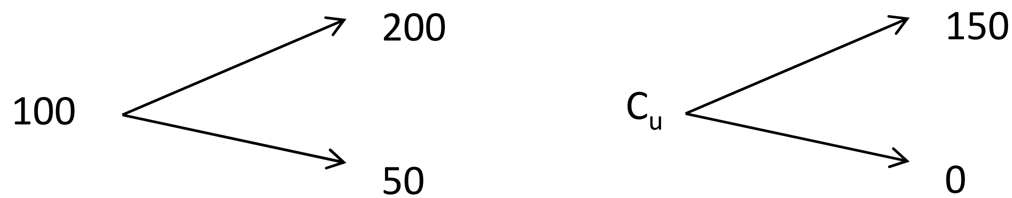


- Option payoff:



Two-period binomial tree

- To price the option, work backwards from final period.



- We know how to price this from before:

$$p = \frac{R-d}{u-d} = \frac{1.25-0.5}{2-0.5} = 0.5$$

$$C_u = \frac{1}{R}\left[pC_{uu} + (1-p)C_{ud}\right] = 60$$

- **Three-step procedure:**
  - 1. Compute risk-neutral probability, p
  - 2. Plug into formula for C at each node to for prices, going backwards from the final node.
  - 3. Plug into formula for Δ and B at each node for replicating strategy, going backwards from the final node..

Two-period binomial tree

- General formulas for two-period tree:
- $p=(R-d)/(u-d)$

$C=[pC_u+(1-p)C_d]/R$
$\quad =[p^2C_{uu}+2p(1-p)C_{ud}+(1-p)^2C_{ud}]/R$
$\Delta=(C_u-C_d)/(uS-dS)$
$B=C- \Delta S$

$C_u=[pC_{uu}+(1-p)C_{ud}]/R$
$\Delta_u=(C_{uu}-C_{ud})/(u^2S-udS)$
$B_u=C_u- \Delta_u S$

$C_d=[pC_{ud}+(1-p)C_{dd}]/R$
$\Delta_d=(C_{ud}-C_{dd})/(udS-d^2S)$
$B_d=C_d- \Delta_d S$

C_{uu}

C_{ud}

C_{dd}

- Synthetic option requires ***dynamic hedging***
  - Must change the portfolio as stock price moves

Arbitraging a mispriced option

- Consider a 3-period tree with S=80, K=80, u=1.5, d=0.5, R=1.1
- Implies p = (R-d)/(u-d) = 0.6
- Can dynamically replicate this option using 3-period binomial tree. Turns out that the cost is $34.08
- If the call is selling for $36, how to arbitrage?
  - Sell the real call
  - Buy the synthetic call
- What do you get up front?
  - $C-\Delta S+B = 36 - 34.08 = 1.92$

Arbitraging a mispriced option

- Suppose that one period goes by (2 periods from expiration), and now S=120. If you close your position, what do you get in the following scenarios?
  - Call price equals "theoretical value", $60.50.
  - Call price is less than 60.50
  - Call price is more than 60.50
- Answer:
  - Closing the position yields zero if call equals theoretical
  - If call price is less than 60.50, closing position yields more than zero since it is cheaper to buy back call.
  - If call price is more than 60.50, closing out position yields a loss! What do you do? (Rebalance and wait.)

Towards Black-Scholes

- Black-Scholes can be viewed as the limit of a binomial tree where the number of periods ***n*** goes to infinity
- Take parameters:

$$u = e^{\sigma\sqrt{T/n}}, d = 1/u = e^{-\sigma\sqrt{T/n}}$$

- Where:
  - n = number of periods in tree
  - T = time to expiration (e.g., measured in years)
  - σ = standard deviation of continuously compounded return
- Also take

$$R = e^{rT/n}$$

Towards Black-Scholes

- General binomial formula for a European call on non-dividend paying stock n periods from expiration:

$$C = \frac{1}{R}\left[\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^{j}(1-p)^{n-j} \max(0, u^{j} d^{n-j} S - K)\right]$$

- Substitute u, d, and R and letting n be very large (hand-waving here), get Black-Scholes:

$$C = SN(d_1) - Ke^{rT} N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln(S/K) + (r + \sigma^2/2)T\right]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Interpreting Black-Scholes

- Note that interpret the trading strategy under the BS formula as

$$\Delta_{call} = N(d_1)$$
$$B_{call} = -Ke^{rT} N(d_2)$$

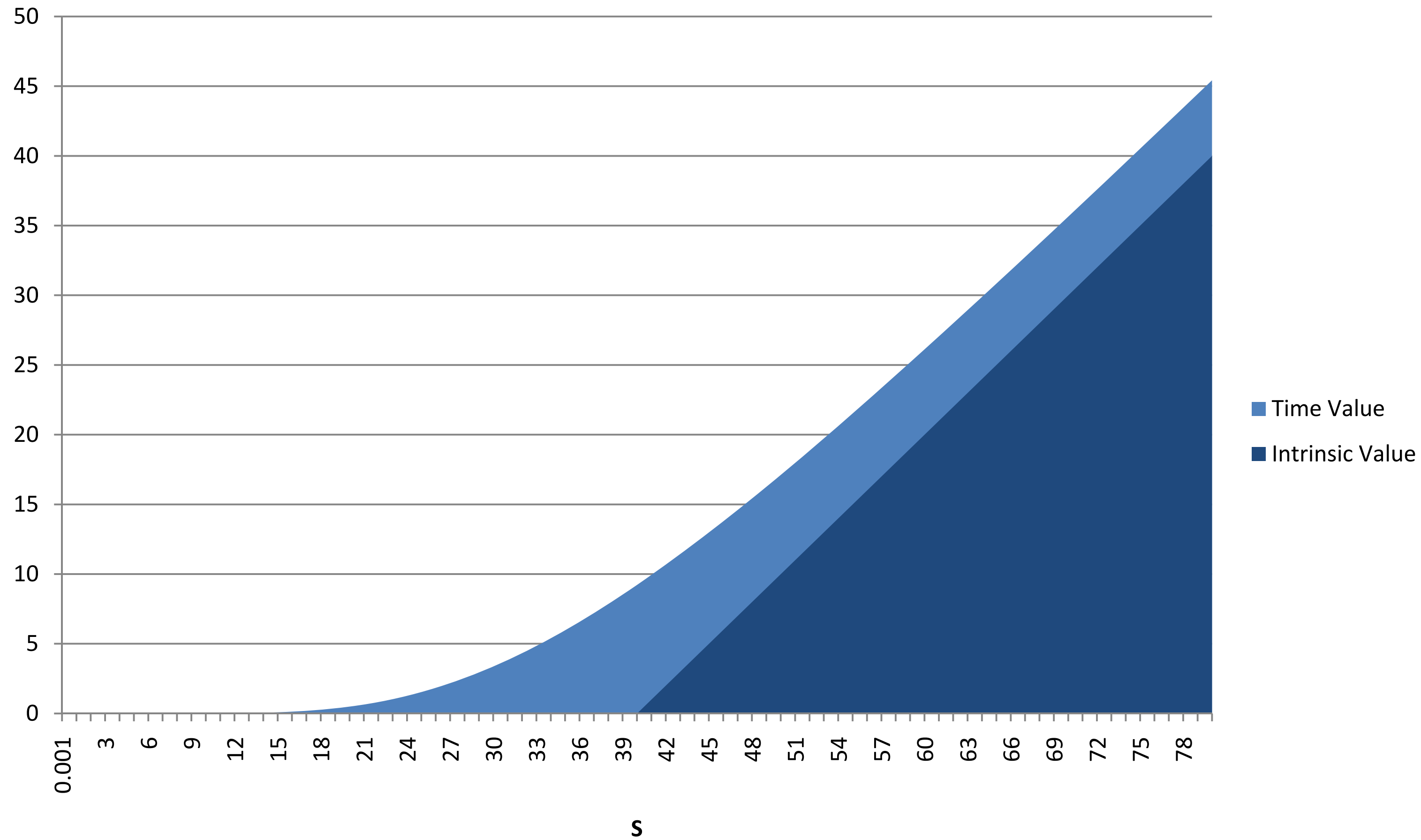
- Price of a put-option: use put-call parity for non-dividend paying stock

$$P = C - S + Ke^{-rT}$$
$$= Ke^{-rT} N(-d_2) - SN(-d_1)$$

- Reminder of parameters
  - 5 parameters
  - **S** = current stock price, **K** = strike, **T** = time to maturity, **r** = annualized continuously compounded risk-free rate, **σ**=annualized standard dev. of cont. compounded rate of return on underlying

Interpreting Black-Scholes

- Option has ***intrinsic value*** [max(S-K,0)] and ***time-value*** [C –max(S-K,0)]



Delta

- Recall that Δ is the sensitivity of option price to a small change in the stock price
  - Number of shares needed to make a synthetic call
  - Also measures riskiness of an option position
- From the formula for a call,

$$\Delta_{call} = N(d_1)$$
$$B_{call} = -Ke^{rT} N(d_2)$$

- A call always has delta between 0 and 1.
- Similar exercise: delta of a put is between -1 and 0.
- Delta of a stock: 1. Delta of a bond: 0.
- Delta of a portfolio: $\Delta_{portfolio} = \sum N_i \Delta_i$

Delta-Hedging

- A portfolio is ***delta-neutral*** if

$$\Delta_{portfolio} = \sum N_i \Delta_i = 0$$

- Delta-neutral portfolios are of interest because they are a way to hedge out the risk of an option (or portfolio of options)
- Example: suppose you ***write*** 1 European call whose delta is 0.61. How can you trade to be delta-neutral?

$$n_c \Delta_{call} + n_s \Delta_S = -1(0.61) + n_s(1) = 0$$

- So we need to hold 0.61 shares of the stock.
- Delta hedging makes you ***directionally neutral*** on the position.

Notes on Black-Scholes

- Delta-hedging is not a perfect hedge if you do not trade continuously
  - Delta-hedging is a linear approximation to the option value
  - But convexity implies second-order derivatives matter
  - Hedge is more effective for smaller price changes
- Delta-Gamma hedging reduces the basis risk of the hedge.
- B-S model assumes that volatility is constant over time. This is a bad assumption
  - Volatility "smile"
  - BS underprices out-of-the-money puts (and thus in-the-money calls)
  - BS overprices out-of-the-money calls (and thus in-the-money puts)
  - Ways forward: stochastic volatility
- Other issues: stochastic interest rates, bid-ask transaction costs, etc.

Implied Vol., Smiles and Smirks

- Implied volatility
  - Use current option price and assume B-S model holds
  - Back out volatility
  - ViX versus implied volatility of 500 stocks
- Smile/Smirk
  - Implied volatility across various strike prices
    - BS implies horizontal line
  - Smile/Smirk after 1987

Collateral debt oblications (CDO)

- Collateralized Debt Obligation- repackage cash flows from a set of assets
- Tranches: Senior tranche is paid out first, Mezzanine second, junior tranche is paid out last
- Can adapt option pricing theory, useful in pricing CDOs:
  - Tranches can be priced using analogues from option pricing formulas
  - Estimate "implied default correlations" that price the tranches correctly