Lecture 11: Multi-period Equilibrium Models

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Time-varying R^*_t (SDF)

- If one-period SDF m_t is not time-varying (i.e. distribution of m_t is i.i.d., then
  - Expectations hypothesis holds
  - Investment opportunity set does not vary
  - Corresponding R* of **single factor** state-price beta model can be easily estimate (because over time one more and more observations about R*)
- If not, then m_t (or corresponding R^*_t)
  - depends on state variable
  - **multiple factor** model

R* depends on state variable

- $R^*_t = R^*(z_t)$, with state variable z_t
- Example:
  - ➢ z_t = 1 or 2 with equal probability
  - ➢ Idea:
    - Take all periods with $z_t=1$ and figure out R*(1)
    - Take all periods with $z_t=2$ and figure out R*(2)
  - ➢ Can one do that?
    - No – hedge across state variables
- Potential state-variables: predict future return

Intertemporal CAPM (ICAPM)

- Merton (1973)

Deriving ICAPM

- ICAPM allows consumption to depend on state variable z_t, which predicts future returns, e.g. price-dividend ratio, risk-free rate

$$E[\sum_{t=0}^{\infty} \delta^t u(c_t, z_t)]$$

- Hence, value function V depends on both wealth W_t and on state variable z
- Bellman equation

$$V(W_t, z_t) = \sup_{c_t, W_t} \{u(c_t, z_t) + \delta E_t[V(W_{t+1}, z_{t+1}]\}$$

Deriving ICAPM

$$V(W_t, z_t) = \sup_{c_t, W_t} \{u(c_t, z_t) + \delta E_t[V(W_{t+1}, z_{t+1}]\}$$

- Recall $W_{t+1} = R^W_{t+1}(W_t - c_t)$.
  Differentiate w.r.t. c_t and W_t

$$0 = u'(c_t, z_t) - \delta E_t[V_W(W_{t+1}, z_{t+1})R^W_{t+1}]$$

$$V_W(W_t, z_t) = \delta E_t[V_W(W_{t+1}, z_{t+1})R^W_{t+1}]$$

- Therefore $u'(c_t, z_t) = V_W(W_t, z_t)$

Deriving ICAPM

- Hence equation

$$E[R^i_{t+1}] - R^f = -\frac{Cov_t(u'(c_{t+1}), R^i_{t+1})}{E_t[u'(c_{t+1})]}$$

becomes
$$E[R^i_{t+1}] - R^f = -\frac{Cov_t(V_W(W_{t+1}, z_{t+1}), R^i_{t+1})}{E_t[V_W(W_{t+1}, z_{t+1})]}$$

- Using a first order approximation

$$V_W(W_{t+1}, z_{t+1}) \approx V_W(W_t, z_t) + V_{WW}(W_t, z_t)\Delta W_{t+1} + V_{Wz}(W_t, z_t)\Delta z_{t+1}$$

we obtain

$$E[R^i_{t+1}] - R^f = -\gamma Cov_t(\Delta W_{t+1}, R^i_{t+1}) + \frac{V_{Wz}}{E_t[V_W]} Cov(\Delta z_{t+1}, R^i_{t+1})$$

- Where γ is relative risk aversion coefficient of V
- Second term are additional "risk factors"

Static problem = intertemporal problem

- In general ICAPM setting
  - CRRA with $\gamma \neq 1$ and changing investment opportunity sets
- Special cases
  1. CRRA and i.i.d. returns and constant r^f
     - SR and LR investors have the same portfolio weights.
     - Solve static problem instead of intertemporal problem
  2. Log utility and non-i.i.d. returns => same result

Digression: Multi-period Portfolio Choice

$$\max_{\{s_t,a_t\}_{t=0}^{T-1}} E[\sum_{t=0}^{T} \delta^t U(c_t)]$$

s.t.

$$c_T = s_{T-1}(1+r_f) + a_{T-1}(r_T - r_f)$$

$$c_t + s_t \leq s_{t-1}(1+r_f) + a_{t-1}(r_t - r_f)$$

$$c_0 + s_0 \leq Y_0$$

Theorem 4.10 (Merton, 1971): Consider the above canonical multi-period consumption-saving-portfolio allocation problem. Suppose U() displays CRRA, r_f is constant and {r} is i.i.d. Then a/s_t is time invariant.

(Dynamic) Hedging Demand

- Illustration with noise trader risk:
  - Suppose fundamental value is constant v=1, but price is noisy (due to noise traders)
  - If the asset is underpriced, e.g. p=.9 , then it might be even more underpriced in the next period
    - Myopic risk-averse investor:
      buy some of the asset and push price towards 1, but not fully
    - Forward-looking risk-averse investor:
      yes, there can be intermediate losses if price declines in next period, but then **investment opportunity set** improves even more   i.e. if returns are bad, then I have great opportunity (dynamic hedge)

Dynamic hedging demand

- Trade-off
  - Low return realization in next period
    - Good since opportunity going forward is high
      - Invest more
    - Bad since marginal utility is high
      - Consume and invest less
  - High return realization in next period ….
- Utility
  - γ> (<)1 first (second) effect dominates
  - γ=1(log-utility) both effects offset each other

Conditional vs. unconditional CAPM

- If β of each subperiod CAPM are time-independent, then
conditional CAPM = unconditional CAPM
- If βs are time-varying they may co-vary with R_m and hence CAPM equation does not hold for unconditional expectations.
  - Additional co-variance terms have to be considered!
  - Move from single-factor setting to multi-factor setting