Fin 501: Asset Pricing
Pricing Models and Derivatives
Problem Set 2

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1.1 Problem (20 points)

Utility function: Under certainty, any increasing monotone transformation of a utility function is also a utility function representing the same preferences. Under uncertainty, we must restrict this statement to linear transformations if we are to keep the same preference representation. Give a mathematical as well as an economic interpretation for this.

Check it with this example. Assume an initial utility function attributes the following values to 3 outcomes:

$$\begin{array}{ll} B & u(B) = 100 \\ M & u(M) = 10 \\ P & u(P) = 50 \end{array}$$

a) **Check that with this initial utility function, the lottery** $L = (B, M, 0.50) \succ P$.

b) **The proposed transformations are** $f(x) = a + bx, a \geq 0, b > 0$**, and** $g(x) = \ln(x)$**. Check that under** f**,** $L \succ P$**, but that under** g**,** $P \succ L$**.**

1.2 Problem (20 points)

Inter-temporal consumption: Consider a two-date (one-period) economy and an agent with utility function over consumption:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

at each period. Define the inter-temporal utility function as $V(c_1, c_2) = U(c_1) + U(c_2)$**. Show (try it mathematically) that the agent will always prefer a smooth consumption stream to a more variable one with the same mean, that is,**

$$U(\bar{c}) + U(\bar{c}) > U(c_1) + U(c_2)$$
$$\text{if } \bar{c} = \frac{c_1 + c_2}{2}, c_1 \neq c_2.$$

1.3 Problem (20 points)

Risk aversion: Consider the following utility functions (defined over wealth Y**):**

$$\begin{aligned}
(1)\ U(Y) &= -\frac{1}{Y} \\
(2)\ U(Y) &= \ln(Y) \\
(3)\ U(Y) &= -Y^{-\gamma} \\
(4)\ U(Y) &= -\exp(-\gamma Y) \\
(5)\ U(Y) &= \frac{Y^{\gamma}}{\gamma} \\
(6)\ U(Y) &= \alpha Y - \beta Y^2
\end{aligned}$$

a) Check that they are well behaved ($U' > 0, U'' < 0$**) or state restrictions on the parameters so that they are (utility functions** $(1) - (5)$**). For utility function** (6)**, take positive** α **and** β**, and give the range of wealth over which the utility function is well behaved.**

b) Compute the absolute and relative risk aversion coefficients.

c) What is the effect of the parameter γ **(when relevant)?**

d) Classify the functions as increasing/decreasing risk aversion utility functions (both absolute and relative).

1.4 Problem (20 points)

Certainty equivalent:

$$\begin{aligned}(1)\ U(Y) &= -\frac{1}{Y}\\(2)\ U(Y) &= \ln(Y)\\(3)\ U(Y) &= \frac{Y^{\gamma}}{\gamma}\end{aligned}$$

Consider the lottery $L_1 = (50,000; 10,000; 0.50)$. Determine the lottery $L_2 = (x; 0; 1)$ that makes an agent indifferent to lottery L_1 with utility functions $(1), (2)$, and (3) as defined. For utility function (3), use $\gamma = \{0.25, 0.75\}$. What is the effect of changing the value of γ? Comment on your results using the notions of risk aversion and certainty equivalent.

1.5 Problem (20 points)

Risk premium: A businesswoman runs a firm worth \$ 100,000. She faces some risk of having a fire that would reduce her net worth according to the following three states, $i = 1, 2, 3$, each with probability $\pi(i)$ (Scenario A).

State	*Worth*	$\pi(i)$
1	1	0.01
2	50,000	0.04
3	100,000	0.95

Of course, in state 3, nothing detrimental happens, and her business retains its value of \$ 100,000.

a) **What is the maximum amount she will pay for insurance if she has a logarithmic utility function over final wealth? (Note: The insurance pays \$ 99,999 in the first case; \$ 50,000 in the second; and nothing in the third.)**

b) **Do the calculations with the following alternative probabilities:**

	Scenario B	*Scenario C*
$\pi(1)$	0.01	0.02
$\pi(2)$	0.05	0.04
$\pi(3)$	0.94	0.94

Is the outcome (the comparative change in the premium) a surprise? Why?

1.6 Problem (20 points)

Risk aversion and portfolio choice: Consider an economy with two types of financial assets: one risk-free and one risky asset. The rate of return offered by the risk-free asset is r_f. The rate of return of the risky asset is $\tilde{r}$. Note that the expected rate of return $E(\tilde{r}) > r_f$.

Agents are risk-averse. Let Y_0 be the initial wealth. The purpose of this exercise is to determine the optimal amount a to be invested in the risky asset as a function of the Arrow-Pratt measure of absolute risk aversion.

The objective of the agents is to maximize the expected utility of terminal wealth:

$$\max_a E(U(Y))$$

where E is the expectation operator, $U(\cdot)$ is the utility function with $U' > 0$ and $U'' < 0$, Y is the wealth at the end of the periond, and a is the amount being invested in the risky asset.

a) Determine the final wealth as a function of a, r_f, and $\tilde{r}$.

b) Compute the f.o.c. (first order condition). Is this a maximum or a minimum?

c) We are interested in determining the sign of da^*/dY_0. Calculate first the total differential of the f.o.c. as a function of a and Y_0. Write the expression for da^*/dY_0. Show that the sign of this expression depends on the sign of its numerator.

d) You know that R_A, the absolute risk aversion coefficient, is equal to $-U''(\cdot)/U'(\cdot)$. What does it mean if $R_A' = dR_A/dY < 0$?

e) Assuming $R_A' < 0$, compare $R_A(Y)$ and $R_A(Y_0(1+r_f))$: Is $R_A(Y) > R_A(Y_0(1+r_f))$ or vice-versa? Don't forget there are two possible cases: $\tilde{r} \geq r_f$ and $\tilde{r} < r_f$.

f) Show that $U''(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f) \geq -R_A(Y_0(1+r_f))$ $\times U'(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)$ for both cases in part e).

g) Finally, compute the expectation of $U''(Y)(\tilde{r} - r_f)$. Using the f.o.c., determine its sign. What can you conclude about the sign of da^*/dY_0? What was the key assumption for the demonstration?