# Solving Heterogeneous-Agent Models with Financial Frictions: A Continuous-Time Approach.

September 13, 2011

**Notes prepared for Yuliy's lectures at "Princeton Initiative: Macro, Money and Finance," based on work with Markus.**

This lecture covers the techniques of solving heterogeneous-agent models with financial frictions in continuous time. There are different types of agents, and financial frictions place restrictions on the agents' asset holdings and risk exposures. For example, it may be difficult for a productive agent to get financing in the form of debt or equity. Because of these restrictions, the wealth distribution across agents matters. Macro shocks affect the wealth distribution in the economy, and thus the agents' asset holdings, productivity, prices and risk premia. Regulation and monetary policy can affect the wealth distribution as well. An equilibrium is a map from any history of macro shocks to the current state of the economy - described by asset prices, asset allocation, and the agents' actions. Such a map is an equilibrium if (1) all agents behave to maximize utility and (2) markets clear. This lecture covers the techniques to translate these two sets of conditions into an equilibrium characterization.

We get a number of benefits from this exercise. First, we simply see how asset prices and the level of economic activity depend on the wealth distribution in the economy. Second, we see how shocks can have persistent effects on the economy, through their impact on the wealth distribution, and how the impact of shocks on the wealth distribution is amplified through prices. Third, the state space will be divided into regions of stability and instability. In unstable regions, as agents become constrained, their demand for assets and prices respond strongly to shocks, resulting high price volatility. Volatility that results from financial frictions, rather than fundamental shocks, is termed *endogenous* risk. Fourth, as agents choose leverage endogenously, they will try to avoid those constrained regions. Thus, in normal times they economy tends to fall into stable regions, but rare large shocks tend to push it into unstable regions.

To introduce the logic of these models gradually, let me start with one particularly tractable version, with special assumptions. I start with this simple model to illustrate how equilibrium conditions - utility maximization and market clearing - translate into an equilibrium characterization. Naturally, the more special assumptions we make, the less we get out of the solution. From the first simple model we will not get any price effects or endogenous risk. However, we will get some interesting takeaways, such as that the risk premia spike up in crises.

### A Simple Model.

This model is borrowed from Basak and Cuoco (1998). The economy has a risky asset in positive net supply, and a risk-free asset in zero net supply. There are two types of agents - experts and households. Only experts can hold the risky asset - households can only lend to experts at the risk-free rate $r_t$, determined endogenously in equilibrium. The friction is that experts can finance their holdings of the risky asset only through debt - by selling short the risk-free asset to households. That is, experts cannot issue equity. We assume that all agents are small, and behave as price-takers. That is, unlike in microstructure models with noise traders, agents have no price impact.

In the aggregate, the risky asset pays dividend

$$\frac{dD_t}{D_t} = g\ dt + \sigma\ dZ_t,$$

where $g$ is the dividend growth rate, and $Z$ is a standard Brownian motion. The price of risky asset is also determined endogenously, and $q_t$ denotes the price-to-dividend ratio. Thus, the aggregate value of all assets in the economy is $q_tD_t$. If $N_t$ is the aggregate net worth of experts, then the aggregate net worth of households is $q_tD_t - N_t$.

For *tractability*, all agents are assumed to have logarithmic utility with discount rate $\rho$, of the form

$$E\left[\int_0^\infty e^{-\rho t}\log c_t\,dt\right],$$

where $c_t$ is consumption at time $t$. Logarithmic utility has two convenient properties, which help reduce the number of equations that characterize equilibrium. First, for agents with log utility

$$\text{consumption} = \rho \cdot \text{net worth} \tag{1}$$

that is, they always consume a fixed fraction of wealth regardless of the risk-free rate or risky investment opportunities. Second, the allocation of wealth between the risky and the risk-free asset is characterized by the equation

$$\text{volatility of wealth} = \text{Sharpe ratio of risky investment}, \tag{2}$$

where the volatility of wealth is measured in percent.[1]

We use equations (1) and (2) to formalize equilibrium conditions, and characterize equilibrium.

**Definition.** *An equilibrium is a map from histories of macro shocks* $\{Z_s,\ s \le t\}$ *to the price of capital* $q_t$*, risk-free rate* $r_t$*, as well as asset holdings and consumption choices of all agents, such that*

1. *agents behave to maximize utility and*

2. *markets clear*

To find an equilibrium, we need to write down equations that processes $q_t$, $r_t$, etc. have to satisfy, and from those, characterize how these processes evolve with the realizations of shocks $Z$. Usually, it is convenient to express this relationship using a state variable, which describes the distribution of wealth. A good state variable to use is the fraction of wealth owned by the experts,

[1]For example, if the annual volatility of S&P 500 is 15% and the risk premium is 3% (so that the Sharpe ratio is 3%/15% = 0.2), then a log utility agent wants to hold a portfolio with volatility 0.2 = 20%. This corresponds to a weight of 1.33 on S&P 500, and -0.33 on the risk-free asset.

$$\eta_t = \frac{N_t}{q_t D_t},$$

which takes values between 0 and 1. When $\eta_t$ drops, experts become more constrained, and so small values of $\eta_t$ correspond to a crisis regime.

So, how can we solve for the equilibrium? In two steps! First, we use the equilibrium conditions, i.e. utility and maximization and market clearing, to write down restrictions $q_t$ and $r_t$ need to satisfy. In this simple model, we will be able to express $q_t$ and $r_t$ as functions of $\eta_t$. Second, we need to derive the law of motion of $\eta_t$.

**Step 1: The Equilibrium Conditions.** First, from condition (1), the aggregate consumption of all agents is $\rho q_t D_t$, and aggregate output is $D_t$. From market clearing for consumption goods, these must be equal, and so

$$q_t = \frac{1}{\rho} \tag{3}$$

is the equilibrium price of the risky asset (per unit flow of dividend). The aggregate consumption of experts must be $\rho N_t = \eta_t D_t$, and the aggregate consumption of households is $(1 - \eta_t) D_t$.

Second, we can use condition (2) for experts to figure out the equilibrium risk-free rate. We look at the return on risky and risk-free assets to compute the Sharpe ratio of risky investments. We look at balance sheets of experts to compute the volatility of their wealth. Then we use equation (2) to get the risk-free rate.

Because $q_t$ is constant, risky asset earns the return of

$$dr_t^D = \underbrace{1/q_t\, dt}_{\rho,\text{ dividend yield}} + \underbrace{g\, dt + \sigma\, dZ_t}_{\text{capital gains rate}},$$

and risk-free asset earns $r_t$ so the Sharpe ratio of risky investment is

$$\frac{\rho + g - r_t}{\sigma}.$$

Because experts must hold all the risky assets in the economy, with value $q_t D_t$ (households cannot hold them), and absorb risk through net worth $N_t$, the volatility of their net worth is

$$\frac{q_t D_t}{N_t}\sigma = \frac{\sigma}{\eta_t}.$$

Using (2),

$$\frac{\sigma}{\eta_t} = \frac{\rho + g - r_t}{\sigma} \quad \Rightarrow \quad r_t = \rho + g - \frac{\sigma^2}{\eta_t}. \tag{4}$$

**Step 2: The Law of Motion of $\eta_t$.** To finish deriving the equilibrium, we need to describe how shocks $Z$ affect the state variable $\eta_t = N_t/(q_t D_t)$. We have, using Ito's lemma

$$dN_t = \underbrace{q_t D_t \, dr_t^D}_{\text{risky investment}} + \underbrace{(N_t - q_t D_t) r_t \, dt}_{\text{risk-free investment}} - \underbrace{\rho N_t \, dt}_{\text{consumption}}, \tag{5}$$

$$dD_t = g D_t \, dt + \sigma D_t \, dZ_t \quad \Rightarrow \quad d\frac{1}{q_t D_t} = (-g + \sigma^2) \frac{1}{q_t D_t} \, dt - \sigma \frac{1}{q_t D_t} \, dZ_t. \tag{6}$$

and so

$$d\eta_t = \frac{1}{q_t D_t} dN_t + N_t d\left(\frac{1}{q_t D_t}\right) + \text{Cov}\left(N_t, \frac{1}{q_t D_t}\right) = \frac{(1-\eta_t)^2}{\eta_t} \sigma^2 dt + (1-\eta_t)\sigma dZ_t. \tag{7}$$

**Observations.** A few observations about what happens in equilibrium. Variable $\eta_t$ fluctuates with macro shocks - a positive shocks increases the wealth allocation of experts, because experts are levered. A negative shock erodes $\eta_t$, and experts require a higher risk premium to hold risky assets. Experts are convinced to keep holding risky assets by the increasing Sharpe ratio

$$\frac{\sigma}{\eta_t} = \frac{\rho + g - r_t}{\sigma},$$

which goes to $\infty$ as $\eta_t$ goes to 0. Strangely, this is achieved due to the risk-free rate $r_t = \rho + g - \sigma^2/\eta_t$ going to $-\infty$, rather than due to depressed price of the risky asset. Because $q_t$ is constant, there is no endogenous risk, no amplification and no volatility effects. Therefore, in this model, assumptions that allow such a simple solution also eliminate any price effects that we are so interested in. We have to work harder to get those effects. [2]

[2]Besides the absence of price effects, another problem with this model is that in the long run expert sector becomes so large that it overwhelms the whole economy. To see this, not that the drift of $\eta_t$ is always positive. This feature is typical of models in which one group of agents has an advantage over another group - in this case only experts can invest in the risky asset. It is possible to prevent expert sector from becoming too large through

However, now at least we have seen how equilibrium conditions can be translated into formulas that describe how the economy behaves. Next, as we consider a more complicated models, in which the price of the risky asset $q_t$ reacts to shocks. We also develop methodology that allows agents to have more complicated preferences, for nontrivial distribution of assets among agents, and for investment.

(discrete vs. continuous time)

### Returns with Investment and Endogenous Risk.

Consider a risky asset, which produces gross cash flow (before investment) of

$$\frac{dD_t}{D_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t, \tag{8}$$

where $\iota_t$ is the percentage of cash flows reinvested. Typically, the investment function is assumed to satisfy $\Phi(0) = 0$, $\Phi' > 0$ and $\Phi'' \leq 0$. Thus, in the absence of investment, the asset simply depreciates at rate $\delta$. The concavity of $\Phi$ reflects decreasing returns to scale, and for negative values of $\iota$, corresponds to *technological illiquidity* - the marginal cost of capital depends on the rate of investment/disinvestment. The optimal rate of investment also depends on the market price of capital $q_t$ (per unit of gross dividend rate), and maximizes $q_t\Phi(\iota_t) - \iota$. The first-order condition

$$q_t\Phi'(\iota) = 1$$

completely determines the optimal rate of investment $\iota(q_t)$, the dividend growth rate $g(q_t) = \Phi(\iota(q_t)) - \delta$ and the net dividend yield of $d(q_t) = \frac{1-\iota(q_t)}{q_t}$.

Suppose that $q_t$ follows the law of motion

$$\frac{dq_t}{q_t} = \mu_t^q\, dt + \sigma_t^q\, dZ_t, \tag{9}$$

an additional assumption. For example, Bernanke, Gertler and Gilchrist (1999) assume that experts are randomly hit by a shock that makes them households. Alternatively, if experts have a higher discount rate than households, then greater consumption rate prevents expert sector from becoming too large.

which, of course, is endogenous in equilibrium. Then, using Ito's lemma, the return from investing in the risky asset is given by

$$dr_t^D = \underbrace{d(q_t)\,dt}_{\text{dividend yield}} + \underbrace{(g(q_t) + \mu_t^q + \sigma\sigma_t^q)\,dt + (\sigma + \sigma_t^q)\,dZ_t}_{\frac{d(q_t D_t)}{q_t D_t}\text{, the capital gains rate}}. \tag{10}$$

Thus, generally a part of the risk from investment is fundamental, $\sigma\,dZ_t$, and a part is endogenous, $\sigma_t^q\,dZ_t$.

### Optimal Portfolio Choice.

Consider an agent, whose marginal utility of wealth $\theta_t$ follows

$$\frac{d\theta_t}{\theta_t} = \mu_t^\theta\,dt + \sigma_t^\theta\,dZ_t. \tag{11}$$

Then the agent's stochastic discount factor (SDF) at time $t$ is $e^{-\rho s}\theta_{t+s}/\theta_t$ for payoff received at time $t + s \geq t$, where $\rho$ is the rate at which the agent discounts utility. The SDF can be used to price assets: for an asset with return

$$dr_t^A = \mu_t^A\,dt + \sigma_t^A\,dZ_t,$$

the following asset-pricing relationship has to hold

$$0 = \mu_t^\theta - \rho + \mu_t^A + \sigma_t^A\sigma_t^\theta. \tag{12}$$

This relationship ensures that if wealth $n_t$ is invested in asset A, so that $dn_t/n_t = dr_t^A$, then $n_{t+s}e^{-\rho s}\theta_{t+s}/\theta_t$ is a martingale. Equation (12) is very important and used often in analysis of continuous-time heterogeneous-agent models.

*Example 1.* Let us see how equation (2) for a log utility agent follows from a more general relationship (12). Note that the agent's marginal utility is $\theta_t = 1/c_t$, where consumption $c_t$ is proportional to net worth according to (1). Therefore, if the volatility of net worth is $\sigma_t^n$, then $\sigma_t^\theta = -\sigma_t^n$. For a risky asset with return $r_t^A$, (12) implies

$$0 = \mu_t^\theta - \rho + \mu_t^A - \sigma_t^A\sigma_t^n. \tag{13}$$

For the risk-free asset, whose volatility is 0,

$$0 = \mu_t^\theta - \rho + r_t. \tag{14}$$

Subtracting (14) from (13), we get

$$\mu_t^A - r_t - \sigma_t^A \sigma_t^n = 0 \quad \Rightarrow \quad \frac{\mu_t^A - r_t}{\sigma_t^A} = \sigma_t^n,$$

where the left hand side is the Sharpe ratio, and the right hand side is the volatility of net worth.

*Example 2.* In general, assets can be priced from consumption of risk-averse agents. Consider an agent with CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

whose consumption follows

$$\frac{dc_t}{c_t} = \mu_t^c \, dt + \sigma_t^c \, dZ_t.$$

Then, by Ito's lemma, marginal utility $c^{-\gamma}$ follows

$$\frac{d(c_t^{-\gamma})}{c_t^{-\gamma}} = \left( -\gamma \mu_t^c + \frac{\gamma(\gamma+1)}{2} (\sigma_t^c)^2 \right) dt - \gamma \sigma_t^c \, dZ_t.$$

Any risky investment with return $dr_t^A$, accessible to this agent, must satisfy the pricing equaiton

$$0 = -\gamma \mu_t^c + \frac{\gamma(\gamma+1)}{2} (\sigma_t^c)^2 - \rho + \mu_t^A - \gamma \sigma_t^c \sigma_t^A.$$

We will not use this relationship in this lecture, but it can be used to solve heterogeneous-agent models with risk aversion, such as that of He and Krishnamurthy (2010).

## A Model with Price Effects and Instabilities.

We now illustrate how these principle can be used to solve a more complex model, which we borrow from Brunnermeier and Sannikov (2011). We will be able to get a number of important takeaways from the model:

1. Equilibrium dynamics is characterized by a relatively stable steady state, where the system spends most of the time, and a crisis regime. In the steady state, experts are adequately capitalized, and they channel

excess profits to payouts. They can easily absorb usual macro shocks by adjusting payouts, and prices near the steady state are quite stable. However, an unusually long sequence of negative shocks causes experts to suffer significant losses, and pushes the equilibrium into a crisis regime. In the crisis regime, experts are undercapitalized and constrained. Shocks affect their demand for assets, and thus affect prices of the assets that experts hold. This creates feedback effects, which cause high endogenous risks.

2. High volatility during crisis times may push the system in a very depressed region, where experts' net worth is close to 0. If that happens, it takes a long time for the economy to recover. Thus, the system spends a considerable amount of time far away from the steady state. The stationary distribution is bimodal.

3. Endogenous risk during crises makes assets more correlated.

4. There is a volatility paradox, because risk-taking is endogenous. If the aggregate risk parameter $\sigma$ becomes smaller, the economy does not become more stable. The reason is that experts allow greater leverage, and pay out profits sooner, in response to lower fundamental risk. Due to greater leverage, the economy is prone to crises even when exogenous shocks are smaller. In fact, endogenous risk during crises may actually be higher when $\sigma$ is lower.

5. Financial innovations, such as securitization and derivatives hedging, that allow for more efficient risk-sharing among experts, may make the system less stable in equilibrium. The reason, again, is that risk-taking is endogenous. By diversifying idiosyncratic risks, experts tend to increase leverage, amplifying systemic risks.

(comparisons with BGG and KM)

The model is as follows. There are two types of agents - experts and households. There is capital in positive net supply, and the risk-free asset in zero net supply. The financial friction is that neither experts nor households can issue equity backed by their asset holdings - they can only borrow through risk-free debt.[3]

[3]Brunnermeier and Sannikov (2011) allow some equity issuance, but here we restrict attention to debt only to simplify exposition.

Experts are more productive at managing capital than households. The experts' production technology is characterized by (8). Capital held by households produces a lower dividend stream of $\underline{a}D_t$ instead of $D_t$, where $\underline{a} \leq 1$, and depreciates at a faster rate $\underline{\delta}$. Also, for simplicity we assume that households cannot invest, i.e. under their management,

$$\frac{dD_t}{D_t} = -\underline{\delta}\, dt + \sigma\, dZ_t.$$

Thus, households earn the return of

$$d\underline{r}_t^D = \underbrace{\underline{a}/q_t\, dt}_{\text{dividend yield}} + \underbrace{(\mu_t^q - \underline{\delta} + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t}_{\frac{d(q_t D_t)}{q_t D_t},\ \text{the capital gains rate}} \tag{15}$$

when they manage capital.

Regarding *preferences*, we assume that both experts and households are risk-neutral, but (1) the experts' discount rate $\rho$ is higher than that of households, $r$, and (2) experts cannot have negative consumption, but households can consume negatively. The second assumption simplifies analysis - it implies that households are always financially unconstrained, and that they are willing to lend and borrow arbitrary amounts at the risk-free rate of $r$. To summarize, experts and household maximize, respectively

$$E\left[\int_0^\infty e^{-\rho t} dc_t\right],\ dc_t \geq 0, \quad \text{and} \quad E\left[\int_0^\infty e^{-rt} d\underline{c}_t\right].$$

We denote the fraction of capital allocated to experts by $\psi_t \leq 1$, and look for an equilibrium. That is, we want to characterize how any history of shocks $\{Z_s, s \leq t\}$ maps to equilibrium prices $q_t$, asset allocation $\psi_t$ and consumption so that (1) all agents maximize utility and (2) markets clear.

We will solve for the equilibrium in three steps. First, we introduce the experts marginal utility of wealth $\theta_t$, and use asset pricing and market clearing conditions to write down equations that stochastic laws of motion of $q_t$, $\theta_t$ and $\psi_t$ must satisfy. Second, we focus on the experts' balance sheets to write down the law of motion of

$$\eta_t = \frac{N_t}{q_t D_t},$$

expert wealth as a percentage of the whole wealth in the economy. Third, we look for a Markov equilibrium, and characterize equations for $q_t$, $\theta_t$ and $\psi_t$ as functions of $\eta_t$. We solve these equations numerically.

**Step 1: The Equilibrium Conditions.** From the asset pricing equation (12), experts price the risk-free asset according to

$$0 = \mu_t^\theta - \rho + r, \tag{16}$$

and capital, with return given by (10), according to

$$0 = \mu_t^\theta - \rho + d(q_t) + g(q_t) + \mu_t^q + \sigma\sigma_t^q + (\sigma + \sigma_t^q)\sigma_t^\theta. \tag{17}$$

In equilibrium, the experts marginal utility of wealth must always satisfy $\theta_t \geq 1$, and experts consume only when $\theta_t = 1$. We will see that in equilibrium experts consume only at one point, when $\eta_t$ reaches a critical level $\eta^*$.

Because households can consume both positive and negative amounts, their marginal utility of wealth is always 1. If the expected return on the risky asset is $r$ according to (15), households are willing to hold some of it, i.e. $\psi_t$ can be less than 1. The expected household return on the risky asset cannot exceed $r$ (otherwise they demand an infinite amount of the risky asset, and markets will not clear), but it can be less than $r$ if $\psi_t = 1$. Thus, we have

$$\underline{a}/q_t + \mu_t^q - \underline{\delta} + \sigma\sigma_t^q \leq r, \quad \text{with equality if } \ \psi_t < 1. \tag{18}$$

We will use three conditions (16), (17) and (18) to characterize $q_t$, $\theta_t$ and $\psi_t$ as functions of $\eta_t$. Before we do that, though, we must derive an equation for the law of motion of $\eta_t = N_t/(q_t D_t)$.

**Step 2: The Law of Motion of $\eta_t$.** The law of motion of $N_t$ in this model is analogous to (5) except that experts invest only wealth $\psi_t q_t D_t$ in capital, and they consume sporadically (only when $\theta_t = 1$). We have

$$dN_t = \psi_t q_t D_t \, dr_t^D - (\psi_t q_t D_t - N_t) r \, dt - dC_t,$$

where $dr_t^D$ is given by (10). Furthermore,

$$\frac{d(q_t D_t)}{q_t D_t} = \underbrace{dr_t^D - d(q_t)\, dt}_{\text{capital gains rate}} - \underbrace{(1-\psi_t)(\underline{\delta} + g(q_t))\, dt}_{\text{adjustment for households}} \quad \Rightarrow$$

$$\frac{d(1/(q_t D_t))}{1/(q_t D_t)} = -dr_t^D + d(q_t)\, dt + (1-\psi_t)(\underline{\delta} + g(q_t))\, dt + (\sigma + \sigma_t^q)^2\, dt.$$

Using Ito's lemma again,

$$d\eta_t = (dN_t)\frac{1}{q_t D_t} + N_t\, d\left(\frac{1}{q_t D_t}\right) + \psi_t q_t D_t (\sigma + \sigma_t^q)\frac{-1}{q_t D_t}(\sigma + \sigma_t^q)\, dt =$$

$$(\psi_t - \eta_t)(dr_t^D - r\,dt - (\sigma + \sigma_t^q)^2\,dt) + \eta_t d(q_t)\,dt + \eta_t(1-\psi_t)(\underline{\delta} + g(q_t)) - d\xi_t,$$

where $d\xi_t = dC_t/(q_t D_t)$.

From (16) and (17) also

$$0 = d(q_t) + g(q_t) + \mu_t^q + \sigma\sigma_t^q - r + (\sigma + \sigma_t^q)\sigma_t^\theta, \tag{19}$$

so

$$dr_t^D - r\,dt = -(\sigma + \sigma_t^q)\sigma_t^\theta\,dt + (\sigma + \sigma_t^q)\,dZ_t,$$

and we can also write

$$d\eta_t = (\psi_t - \eta_t)(\sigma + \sigma_t^q)\left(dZ_t - (\sigma + \sigma_t^q + \sigma_t^\theta)\,dt\right) + \eta_t d(q_t)dt + \eta_t(1-\psi_t)(\underline{\delta} + g(q_t)) - d\xi_t. \tag{20}$$

**Step 3: Converting the equilibrium conditions (16), (17) and (18) and the law of motion (20) into equations for** $q(\eta)$, $\theta(\eta)$ **and** $\psi(\eta)$. This step is completely mechanical. We would like to convert equations (16), (17) and (18) into differential equations for $q(\eta)$, $\theta(\eta)$ and $\psi(\eta)$ through multiple applications of Ito's lemma. Ito's lemma allows us to replace terms such as $\sigma_t^q$, $\mu_t^\theta$, etc. with expressions containing the derivatives of $q$ and $\theta$. Before we dive into this somewhat lengthy but completely mechanical algebra exercise, let me give a very simple and well-known example to illustrate the gist of what we have to do.

*Example 3.* This example is from the well-known endogenous default model of Leland (1994). Equity holders are sitting on assets whose value follows a geometric Brownian motion

$$\frac{dV_t}{V_t} = r\,dt + \sigma\,dZ_t \tag{21}$$

under the risk-neutral measure. Default happens when the value of assets falls to some value of $V_B$ (which is later endogenized). Before default, equity holders must be paying coupons to debt holders at rate $C$. In the event of default, equity holders abandon the assets, and debt holders receive the liquidating value of assets of $\alpha V_B$, where $\alpha \in (0,1)$.

Under the risk-neutral measure, the expected return of any security must be $r$. Thus, if equity $E_t$ follows $dE_t = \mu_t^E E_t\,dt + \sigma_t^E E_t\,dZ_t$, then we must have[4]

$$r = \mu_t^E - C/E_t. \tag{22}$$

[4] Unlike in Leland (1994), I assumed here that there are no taxes, so equity holders do not get any tax shield benefits by paying coupons.

That is, after paying coupons equity holders must receive an expected return of $r$.

Suppose we would like to calculate how the value of equity $E_t$ depends on the value of assets $V_t$. Then we are face a problem that is completely analogous to that of Brunnermeier and Sannikov (2011) model: We have a law of motion of the state variable $V_t$ and a relationship (22) that the stochastic motion of $E_t$ has to satisfy, and we would like to characterize $E_t$ as a function of $V_t$.

How can we do this? Easy. Using Ito's lemma

$$\mu_t^E E_t = rV_t E'(V_t) + \frac{1}{2}\sigma^2 V_t^2 E''(V_t),$$

and so (22) becomes

$$r = \frac{rVE'(V) + \frac{1}{2}\sigma^2 V^2 E''(V)}{E(V)} - \frac{C}{E(V)}. \tag{23}$$

If function $E(V)$ satisfies this equation, then the process $E_t = E(V_t)$ will satisfy (22). We are able to go from an equation like (22) to a differential equation (23) by assuming that the value of equity is a *function* of the value of assets.

We can solve the second-order ordinary differential equation (ODE) (23) if we have two boundary conditions. The relevant boundary conditions in the context of the Leland (1994) model are $E(V_B) = 0$ and that $V - E(V) \to C/r$ as $V \to \infty$.

Our problem is similar to that of Leland (1994): we have an equation for the stochastic law of motion of the state variable (20), as well as conditions (16), (17) and (18) that processes $q_t$, $\theta_t$ and $\psi_t$ must satisfy. Certainly, the equations are more complicated than those of Leland (1994), and the law of motion of $\eta_t$ is endogenous (i.e. it depends on $q_t$, $\theta_t$ and $\psi_t$). However, the mechanics of solving these equations is the same - use Ito's lemma. The goal of the algebra below is, assuming that we know $\eta$, $q(\eta)$, $q'(\eta)$, $\theta(\eta)$, $\theta'(\eta)$ and a guess of $\psi(\eta)$, to compute $q''(\eta)$ and $\theta''(\eta)$, and have a way of checking whether the guess of $\psi(\eta)$ was correct or not.

First, we express $\sigma_t^q$, $\sigma_t^\theta$ and $\sigma_t^\eta$ from what we know:

$$\sigma_t^q q(\eta) = q'(\eta)(\psi_t - \eta_t)(\sigma + \sigma_t^q) \quad \Rightarrow \quad \sigma_t^q = \frac{q'(\eta)}{q(\eta)} \underbrace{\frac{(\psi_t - \eta_t)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\psi_t - \eta_t)}}_{\text{volatility of } \eta_t}, \tag{24}$$

$$\sigma_t^\theta = \frac{\theta'(\eta)}{\theta(\eta)} \qquad \underbrace{\frac{(\psi_t - \eta_t)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\psi_t - \eta_t)}}_{\text{volatility of } \eta_t,\ \sigma_t^\eta \eta = (\psi_t - \eta_t)(\sigma + \sigma_t^q)} \tag{25}$$

Second, using (16) and (17), we can express $\mu_t^q$ and $\mu_t^\theta$ from what we know. Specifically, using (19)

$$\mu_t^q = r - d(q_t) - g(q_t) - \sigma\sigma_t^q - (\sigma + \sigma_t^q)\sigma_t^\theta \tag{26}$$

and using (16)

$$\mu_t^\theta = \rho - r. \tag{27}$$

Third, from (20) the drift of $\eta_t$ is

$$\mu_t^\eta \eta = -\underbrace{(\psi_t - \eta_t)(\sigma + \sigma_t^q)}_{\sigma_t^\eta \eta}(\sigma + \sigma_t^q + \sigma_t^\theta) + \eta_t d(q_t)\,dt + \eta_t(1 - \psi_t)(\underline{\delta} + g(q_t)), \tag{28}$$

and using Ito's lemma,

$$\mu_t^q q(\eta) = \mu_t^\eta \eta q'(\eta) + \frac{1}{2}(\sigma_t^\eta)^2\eta^2 q''(\eta), \quad \mu_t^\theta \theta(\eta) = \mu_t^\eta \eta \theta'(\eta) + \frac{1}{2}(\sigma_t^\eta)^2\eta^2\theta''(\eta).$$

We can now use these expressions to express $q''(\eta)$ and $\theta''(\eta)$ as functions of what we already computed:

$$q''(\eta) = 2\frac{\mu_t^q q(\eta) - \mu_t^\eta \eta q'(\eta)}{(\sigma_t^\eta)^2\eta^2} \quad \text{and} \quad \theta''(\eta) = 2\frac{\mu_t^\theta \theta(\eta) - \mu_t^\eta \eta \theta'(\eta)}{(\sigma_t^\eta)^2\eta^2}. \tag{29}$$

We can check whether our guess of $\psi_t$ is correct from (18). If

$$\underline{a}/q_t + \mu_t^q - \underline{\delta} + \sigma\sigma_t^q > r \tag{30}$$

for our guess of $\psi_t$, we need to adjust $\psi_t$ downward, because households would like to hold more capital. If

$$\underline{a}/q_t + \mu_t^q - \underline{\delta} + \sigma\sigma_t^q < r,$$

we adjust our guess of $\psi_t$ upward, unless already $\psi_t = 1$. We must look for an appropriate value of $\psi_t$ in the interval $(\eta, \min(1, q(\eta)/q'(\eta) + \eta)]$. Indeed, if $\psi_t = \eta_t$, then $\sigma_t^\eta = \sigma_t^\theta = 0$, experts demand no risk premium and their demand for assets clearly exceeds that of households. If $q(\eta)/q'(\eta) + \eta < 1$, then as $\psi \to q(\eta)/q'(\eta) + \eta$ from below, endogenous risk increases without a bound, leading experts to wish to hold no risky assets due to the precautionary motive.

We summarize our algorithm for computing $q''(\eta)$ and $\theta''(\eta)$ from $\eta$, $q(\eta)$, $q'(\eta)$, $\theta(\eta)$, $\theta'(\eta)$ as follows:

**Algorithm 1.** *Start with* $\eta$, $q(\eta)$, $q'(\eta)$, $\theta(\eta)$, $\theta'(\eta)$. *Search for an appropriate value of* $\psi \in (\eta, \min(1, q(\eta)/q'(\eta) + \eta)]$ *via the following procedure.*

*Set* $\psi_L = \eta$ *and* $\psi_H = \min(1, q(\eta)/q'(\eta) + \eta)$. *Repeat the following loop 30 times. Guess* $\psi = (\psi_L + \psi_H)/2$. *Calculate* $\sigma_q$, $\sigma_\eta$, $\sigma_\theta$ *and* $\mu_t^q$ *from (24), (25) and (26). If (30), then adjust the guess of* $\psi$ *down by setting* $\psi_H = \psi$, *else adjust the guess of* $\psi$ *up, by setting* $\psi_L = \psi$.[5]

*After we found an appropriate value of* $\psi$, *we are on the finishing line: we just need to execute (27), (28) and (29) to find* $q''(\eta)$ *and* $\theta''(\eta)$.

Matlab function fnct.m provided with these notes implements this algorithm.

**Boundary Conditions.** We have characterized an the equilibrium via a system of two second-order differential equations for $q(\eta)$ and $\theta(\eta)$. In equilibrium $q(\eta)$ has to be an increasing function: the price of capital increases when the more productive experts are less constrained. At the same time $\theta(\eta)$ is a decreasing function: experts have a higher marginal utility of wealth when other experts are constrained, i.e. $\eta_t$ is low, and asset prices are depressed. As a result, $\theta(\eta)$ must decrease towards 1 over the interval $[0, \eta^*)$ such that $\theta(\eta^*) = 1$. Thus, $d\xi_t = 0$ on $[0, \eta^*)$ (i.e. experts refrain from consumption) and the process $\xi_t$ makes $\eta_t$ to reflect at $\eta^*$, where experts consume.

We need five boundary conditions to solve two second-order equations for $q(\eta)$ and $\theta(\eta)$, and also pin down $\eta^*$. Since $\eta^*$ is a reflecting boundary, we need

$$q'(\eta^*) = \theta'(\eta^*) = 1.$$

Also,

$$\theta(\eta^*) = 1 \quad \text{and} \quad q(0) = \frac{\underline{a}}{r + \underline{\delta}},$$

since household value capital at the price of $\underline{a}/(r + \underline{\delta})$ if experts are wiped out. Finally,

$$\lim_{\eta \to 0} \theta(\eta) = \infty. \tag{31}$$

**Solving the system of ODE's numerically.** We can use function fnct.m together with an ODE solver in Matlab, such as ode45, to solve the

[5] Alternatively, it is also possible to solve a quadratic equation for $\psi$ in this model directly. Also, in order to speed up the code a little bit, one could try $\psi = 1$ first, if that choice is available.

system of equations. We need to perform a search, since our boundary conditions are defined at two endpoints of $[0, \eta^*]$, and we also need to deal with a singularity at $\eta = 0$. The following algorithm performs an appropriate search and deals with the singularity issue, effectively, by solving the system of equations with the boundary condition $\theta(0) = M$, for a large constant $M$, instead of (31):

**Algorithm 2.** *Set* $q(0) = a/(r + \underline{\delta})$, $\theta(0) = 1$ *and* $\theta'(0) = -10^{10}$. *Perform the following procedure to find an appropriate boundary condition* $q'(0)$.

*Set* $q_L = 0$ *and* $q_H = 10^{15}$. *Repeat the following loop 50 times. Guess* $q'(0) = (q_L + q_H)/2$. *Use Matlab function ode45 to solve for* $q(\eta)$ *and* $\theta(\eta)$ *on the interval* $[0, ?)$ *until one of the following events is triggered, either (1)* $q(\eta)$ *reaches the upper bound* $q_{\max} = \max_i \frac{1-i}{r+\delta-\Phi(i)}$, *(2) the slope* $\theta'(\eta)$ *reaches 0 or (3) the slope* $q'(\eta)$ *reaches 0. If integration has terminated for reason (3), we need to increase the initial guess of* $q'(0)$ *by setting* $q_L = q'(0)$. *Otherwise, we decrease the initial guess of* $q'(0)$, *by setting* $q_H = q'(0)$.

*At the end,* $\theta'(0)$ *and* $q'(0)$ *reach 0 at about the same point, which we denote by* $\eta^*$. *Divide the entire function* $\theta$ *by* $\theta(\eta^*)$.[6] *Then plot the solutions.*

Script solve_equilibrium.m provided with these notes implements this algorithm, and uses event function evntfct.m to terminate integration. The solution is economically meaningful even with the boundary condition $\theta(0) = M$: it corresponds to an assumption that, in the event all experts are wiped out, any measure-zero set of experts that still has wealth left gets utility $M$ per dollar of net worth.

**Properties of the Solution.** Point $\eta^*$ plays the role of the steady state of our system. The drift of $\eta_t$ is positive everywhere on the interval $[0, \eta^*)$, because the expert sector, which is more productive than the household sector, is growing in expectation. Thus, the system is pushed towards $\eta^*$ by the drift.

It turns out that the steady state is relatively stable, because volatility is low near $\eta^*$. To see this, recall that the amount of endogenous risk in asset

[6]We can do this because whenever functions $\theta$ and $q$ satisfy our system of equation, so do functions $\Theta\theta$ and $q$ for any constant $\Theta$. Because of that, also, it is immaterial what we set $\theta(0)$ to.

prices, from (24), is given by

$$\sigma_t^q = \frac{q'(\eta)}{q(\eta)} \frac{(\psi_t - \eta_t)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\psi_t - \eta_t)}.$$

From the boundary conditions, $q'(\eta^*) = 0$, so there is no endogenous risk near $\eta^*$.

However, below $\eta^*$, endogenous risk increases as $q'(\eta)$ becomes larger. As prices react to shocks, fundamental risk becomes amplified. As we see from the expression for $\sigma_t^q$, this amplification effect is nonlinear, since $q'(\eta)$ enters not only the numerator, but also denominator. This happens due to the feedback effect: an initial shock causes $\eta_t$ to drop, which leads to a drop in $q_t$, which hurts experts who are holding capital and leads to a further decrease in $\eta_t$, and so on.

Of course, far in the depressed region the volatility of $\eta_t$, $\sigma_t^\eta \eta_t$, becomes low again in this model. This leads to a bimodal stationary distribution of $\eta_t$ in equilibrium.[7]

One can make interesting observations by experimenting with the level of exogenous risk $\sigma$. It would be natural to guess that as $\sigma$ becomes smaller, the economy becomes more stable. This is not necessarily true because experts choose leverage endogenously - that is - they choose endogenously the point $\eta^*$ at which they pay out excess profits. In fact, as $\sigma$ decreases, $\eta^*$ also decreases in equilibrium, i.e. experts pay out profits sooner. While the percent volatility of the risky asset $\sigma + \sigma_t^q$ at the steady state decreases, the maximal volatility of the risky asset below the steady state may in fact become larger. We call this phenomenon the "volatility paradox" - as exogenous risk decreases, higher leverage in equilibrium may in fact lead to more volatile crisis episodes.

[7]Of course, one has to do more to prove that the stationary distribution is bimodal - that involves the asymptotic analysis of the solutions $q$ and $\theta$ near $\eta = 0$, as well as Kolmogorov forward equations that characterize the stationary distribution.