Exercises, "Princeton Initiative: Macro, Money and Finance."

These exercises are provided to help you get comfortable with continuous-time methods of modeling heterogeneous-agent economies. They are based on my lecture at the Princeton Initiative. These are variations of the models from the lecture, and should be solvable with methods we discussed. If you would like to try them and e-mail your solutions to sannikov@gmail.com by Friday September 16, I will take a look at them and e-mail you my own solutions. Please feel free to also consult the notes posted on the Princeton Initiative website. -Yuliy

Problem 1. Consider a version of the Basak and Cuoco (1998) model described in the lecture notes, with one modification. Suppose that experts discount utility at rate $\rho > r$, while households discount utility at rate r. Please answer the following questions. You may use any expressions already derived in the notes.

(a) Using the market-clearing condition, derive the price of capital $q(\eta_t)$.

(b) Assuming that the percent volatility of q_t is σ_t^q, what is the volatility of the return on capital? What is the volatility of expert net worth? What is the Sharpe ratio of the risky investment, implied by experts' portfolio optimization? Derive an expression for $dr_t^D - r_t\,dt$ in terms of σ_t^q.

(c) Derive the law of motion of η_t in equilibrium, in terms of σ_t^q, η_t and the parameters of the model.

(d) Derive an explicit equation for σ_t^q, in terms of η_t and model parameters.

(e) Derive an equation that characterizes steady state(s) of the system, i.e. point(s) η^* where the drift of η_t is zero. For any steady state η^* that is attracting (i.e. the drift points towards η^* around that point), is η^* increasing or decreasing in σ^2?

(f) Derive μ_t^q, as well as the risk-free rate r_t.

Problem 2. Consider a version of the Basak and Cuoco (1998) model described in the lecture notes, with one modification. Suppose that households can also buy capital, but if they do, they earn a lower dividend by a factor of $\underline{a} \in (0, 1)$. Please answer the following questions. You may use any expressions already derived in the notes. (Technical note: It is not essential for answering the following questions, but for this version of the model you need to assume that the risk-free return takes a more general form dR_t rather than $r_t\,dt$. The reason is that a sum invested in the risk-free asset may not evolve in a differentiable manner as the economy transitions between the two regimes described below.)

(a) The equilibrium may feature a regime in which only experts hold capital. What are the price of capital, the risk-free rate and the law of motion of η_t in this regime? What other condition needs to be satisfied, in order for households to refrain from holding capital?

(b) Consider a regime in which experts hold a fraction $\psi_t < 1$ of assets, and households hold fraction $1 - \psi_t$. Write down three equations that characterize the price of the risky asset q_t, the risky-free rate r_t and ψ_t in this regime in equilibrium. These three equations are (1) experts' optimal portfolio choice (2) households' optimal portfolio choice and (3) market clearing, taking into account that each agent consumes ρ times his wealth.

(c) Express the law of motion of η_t in terms of η_t, ψ_t, q_t and σ_t^q.

(d) Express $\sigma + \sigma_t^q$ in terms of $q(\eta)$, $q'(\eta)$, ψ_t and the parameters of the model.

(e) Derive a differential equation that $q(\eta)$ must satisfy in the regime where households invest.

(f) Write Matlab code that solves the differential equation you derived in part (e) and computes the equilibrium. Hint: One boundary condition is $q(0) = \underline{a}/\rho$, and the second boundary condition is that $q(\tilde{\eta})$ coincides with the level of price from part (a) at point $\tilde{\eta}$ where the regime switch occurs. You can find the appropriate function by either searching for $\tilde{\eta}$ or picking an appropriate starting value near $q(0) = \underline{a}/\rho$ (but be careful with the singularity at 0).