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Fiscal Policy over the Real Business Cycle: A Positive Theory*

Abstract

This paper develops and assesses the implications of the political economy model of Battaglini and Coate (2008) for the behavior of fiscal policy over the business cycle. The model predicts that fiscal policy is counter-cyclical with debt increasing in recessions and decreasing in booms. Public spending increases in booms and decreases during recessions, while tax rates decrease during booms and increase in recessions. In both booms and recessions, fiscal policies are set so that the marginal cost of public funds obeys a submartingale. The quantitative implications of the model are assessed by calibrating the model to the U.S. economy using data from 1979 to 2009. Despite its parsimonious structure, the model matches well the empirical distribution of debt and also its high volatility, strong persistence, and negative correlation with output. Consistent with the data, the model implies that public spending and tax rates are persistent and not very volatile.

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1 Introduction

Real business cycle theory develops the idea that business cycles can be generated by random fluctuations in productivity. At the core of this research program, the fundamental issues are how individuals react to productivity shocks and how these reactions affect the macro economy. While the issue of reaction to shocks is typically studied at the *individual level*, it can also be raised at the *societal level*. How do individuals, through their political institutions, collectively decide to adjust fiscal policies in response to changes in productivity? Moreover, what is the role of changes in fiscal policy in amplifying or dampening shocks? Though understanding individual responses to shocks can be addressed with the tools of basic microeconomics, understanding societal responses requires a study of how collective choices are made in complex dynamic environments.

In the last two decades, political economy has made important progress, both theoretically and empirically, in understanding how governments function and the type of distortions that the political process generates in an economy. This *first generation* of research, however, has largely focused on static or two period models that are not well suited to answer the questions raised by real business cycle theory. When longer time horizons are considered, other important elements of the environment (such as shocks, rational forward looking agents, etc) are muted. Thus, the basic question as to how governments react to business cycles is not well understood. Because of this, empirical analysis of the cyclical behavior of fiscal policy remains largely guided by normative models of policy making.

As part of a *second generation* of political economy research analyzing more general dynamic models, Battaglini and Coate (2008) propose a positive theory of fiscal policy.[1] Their framework begins with a tax smoothing model of fiscal policy of the form studied by Barro (1979), Lucas and Stokey (1983), and Aiyagari et al. (2002). The need for tax smoothing is generated by shocks in the benefits of public spending created by events like wars and natural disasters. Politics is introduced by assuming that policy choices are made by a legislature rather than a benevolent planner. Moreover, the framework incorporates the friction that legislators can redistribute tax revenues back to their districts via pork-barrel spending. The theory yields clean predictions on how fiscal policy responds to public spending shocks and provides a sharp account of how politics

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[1] Other examples of this type of work include Acemoglu, Golosov, and Tsyvinski (2008), Azzimonti (2010), Hassler et al (2003), Hassler et al (2005), Krusell and Rios-Rull (1999), Song, Zilibotti, and Storesletten (2010), and Yared (2010).

distorts economic policy-making.

This paper develops and assesses the implications of the Battaglini-Coate theory for the behavior of fiscal policy over the business cycle. To develop these implications, public spending shocks are replaced with revenue shocks generated by random fluctuations in the economy's productivity. Further, these productivity shocks are assumed persistent as opposed to independent and identically distributed. Persistent shocks are essential to capture the implications of cyclical fluctuations. When an economy enters a boom or a recession, legislators' expectations about future tax revenues will clearly be influenced and these changed expectations will impact current taxing, spending, and borrowing decisions. To assess the implications of the theory, the model is calibrated to the U.S. economy using data from the last 30 years. The performance of the model in explaining the debt distribution and the cyclical behavior of fiscal variables is analyzed.

The specific model analyzed assumes that a single good is produced using labor. This good can be consumed or used to produce a public good. Labor productivity follows a two state, serially-correlated Markov process. When productivity is high, the economy is in a "boom" and, when it is low, a "recession". Policy choices in each period are made by a legislature comprised of representatives elected by single-member, geographically-defined districts. The legislature can raise revenues by taxing labor income and by issuing one period risk-free bonds. Public revenues are used to finance public good provision and pork-barrel spending. The legislature makes policy decisions by majority (or super-majority) rule and legislative policy-making is modelled as non-cooperative bargaining.

While the incorporation of persistent shocks complicates the characterization of equilibrium, the model remains tractable. Equilibrium fiscal policies converge to a stochastic steady state in which they vary predictably over the business cycle. Upon entering a boom, public spending will increase and tax rates will fall. Over the course of the boom, public spending will continue to increase until it reaches a ceiling level, and tax rates will decrease until they reach floor levels. When the economy enters a recession, public spending will decrease and tax rates will increase. As the recession progresses, public spending will continue to decrease and tax rates will continue to increase. The overall fiscal stance as measured by the long run pattern of debt is counter-cyclical: government debt decreases in booms and increases in recessions.[2]

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[2] There are a number of definitions of "counter-cyclical" fiscal policy in the literature. Consistent with a Keynesian perspective, Kaminsky, Reinhart, and Vegh (2004) and Talvi and Vegh (2005) define fiscal policy to be

Perhaps the most interesting feature of the cyclical behavior of fiscal policy is that debt *falls* when the economy enters a boom. Intuitively, one might have guessed just the opposite. A boom will increase the expectation of future tax revenues and this may lead legislators to increase borrowing so they can appropriate these extra revenues for their districts. Indeed, this is precisely the logic of the well-known "voracity effect" of Tornell and Lane (1999). This intuition is correct, but ignores the fact that any increase in debt will have permanent effects. Thus, such a "voracity effect"-style debt expansion can arise the first time the economy moves from recession to boom, but, once this happens, the level of debt is too high for it to occur again.

In addition, the paper identifies an interesting implication of the theory concerning the dynamic evolution of the so-called marginal cost of public funds (MCPF). The MCPF, a basic concept in public finance, is the social marginal cost of raising an additional unit of tax revenue. It takes into account the distortionary costs of taxation for the economy. In the model, it depends upon the tax rate and the elasticity of labor supply. The theory implies that, at each point in time and over all phases of the cycle, the equilibrium choice of fiscal policies is such that the MCPF obeys a submartingale.[3] This means the expected MCPF next period is always at least as large as the current MCPF and is sometimes strictly larger. This prediction contrasts with that emerging from a planning model which implies that the MCPF obeys a martingale. Political distortions therefore create a wedge between the current MCPF and the future MCPF.[4]

The model is calibrated in the real business cycle tradition pioneered by Kydland and Prescott (1982). The productivity process and other parameters are chosen to match the empirically observed variation in output, the frequency and length of recessions, and the average debt/GDP and spending/GDP ratios. Despite its simple structure, in particular the two-state technology

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counter-cyclical if government spending rises in recessions and tax rates fall. Adopting a neoclassical perspective, Alesina, Campante, and Tabellini (2008) define as counter-cyclical "a policy that follows the tax smoothing principle of holding constant tax rates and discretionary spending as a fraction of GDP over the cycle". Our definition is that fiscal policy is counter-cyclical if debt falls in booms and rises in recessions. Like Alesina, Campante, and Tabellini, our definition is motivated by tax smoothing principles. However, it recognizes the fact that in a world with incomplete markets and unanticipated productivity shocks, these principles do not imply constant tax rates or government spending over the cycle. While reflecting a neoclassical perspective, our definition does not discriminate between a neoclassical and Keynesian view of optimal fiscal policy over the cycle: in both cases, government debt will rise in recessions and fall in booms. As suggested by Kaminsky, Reinhart, and Vegh (2004), the way to discriminate between these views is to look at the behavior of tax rates and public spending. We will discuss this point in greater detail below.

[3] In our model the assumptions of the standard submartingale convergence theorem are not satisfied, so the MCPF does not converge to a constant or to infinity as $t \to \infty$. Indeed, we show that in the long run the MCPF will have a non degenerate stationary distribution.

[4] This submartingale result is true even in economies without persistent shocks, though this was not noted in Battaglini and Coate (2008).

process, the model produces a distribution of debt which is close to the empirically observed one. This means that debt averages around 40% of GDP, is volatile and is strongly negatively correlated with output. Both in the model and in the data, the tax rate and public spending are much less volatile than debt. However, in the model both taxes and public spending are less volatile than in the data. All fiscal variables have high autocorrelations, which is consistent with the data.

2 Related literature

There is a large literature on the cyclical behavior of fiscal policy, with both theoretical and empirical branches. The benchmark theoretical model used in the literature is the tax smoothing model with perfect foresight (Barro (1979)). In this model, perfectly anticipated cyclical variation in the economy generates fluctuations in tax revenues. The government smooths tax rates and public spending by borrowing in recessions and repaying in booms (see, for example, Talvi and Vegh (2005)). Thus, debt is negatively correlated with changes in GDP, while public spending and tax rates are uncorrelated with changes in GDP.

Support for the predictions of this model comes from Barro (1986) who studies the correlation between debt and income changes for the U.S. federal government. Using data from the period 1916-1982, he finds a negative correlation between changes in debt and changes in GNP.[5] Studies of the correlation between public spending and GDP provide more mixed support.[6] The basic findings are that public spending tends to be slightly pro-cyclical for developed economies, and much more pro-cyclical for developing countries.[7] These findings have been interpreted as suggesting that fiscal policy is basically consistent with the perfect foresight tax smoothing model in developed countries and inconsistent in developing countries.

A variety of theories have been advanced to explain the stronger pro-cyclical behavior of gov-

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[5] Barro runs regressions of the form $(b_t - b_{t-1})/y_t = \alpha \cdot X_t + \beta yvar_t + \varepsilon_t$,where b_t is debt, y_t is GNP, X_t is a vector of control variables, $yvar_t$ is a business cycle indicator, and ε_t is a shock. The business cycle indicator takes on negative values during a boom and positive values during a recession. He finds that the coefficient β is positive, suggesting that debt behaves counter-cyclically.

[6] The correlation between government consumption (which excludes transfers and debt interest payments) and changes in GDP has been studied extensively for the U.S. both at the federal and state level, and for different groups of countries aggregated according to geographical location and stage of economic development. Gavin and Perotti (1997) compare a sample of Latin American countries with a sample of industrialized countries. Sorensen, Wu, and Yosha (2001) study the U.S. states. Lane (2003) looks at all the OECD countries. Alesina, Campante, and Tabellini (2008), Kaminsky, Reinhart, and Vegh (2004), Talvi and Vegh (2005), and Woo (2009) look at data sets containing a broad sample of developed and developing countries.

[7] See, in particular, Alesina, Campante, and Tabellini (2008), Gavin and Perotti (1997), Kaminsky, Reinhart, and Vegh (2004), Talvi and Vegh (2005), and Woo (2009).

ernment spending in developing countries. In an early attempt to explain the phenomenon, Gavin and Perotti (1997) note that pro-cyclical policies may be induced by tighter debt constraints in recessions. Borrowing limits in recessions would force contractionary policies; as the limits are relaxed in booms, we would observe expansionary policies. Other authors point to the dysfunctional political systems that pervade developing countries. In a dynamic common pool framework in which multiple groups compete for a share of the national pie, Lane and Tornell (1998) and Tornell and Lane (1999) suggest that group competition can increase following a positive income shock which may lead spending to increase more than proportionally to the increase in income - the *voracity effect*. In the context of a perfect foresight tax smoothing model, Talvi and Vegh (2005) show that if spending pressures increase with the size of the primary surplus, then optimal fiscal policy will imply a pro-cyclical pattern of spending. In a political agency framework, Alesina, Campante and Tabellini (2008) show that when faced with corrupt governments whose debt and consumption choices are hard to observe, citizens may rationally demand higher public spending in a boom.

The theory presented here is complementary to the political economy theories of Lane and Tornell and Alesina, Campante, and Tabellini. They are interested in modelling different, and much more dysfunctional, political systems than us. As noted in the introduction, our theory predicts that a voracity effect-style debt expansion can arise the first time the economy moves from recession to boom. However, our theory differs from Lane and Tornell's work in that our economy is subject to recurrent cyclical shocks rather than a one time permanent shock that is either unforeseen or perfectly anticipated at time zero. This accounts for our conclusions that the voracity effect does not arise in the long run.

A deeper problem with the perfect foresight tax smoothing model is that its predictions are not robust to relaxing the assumption of perfect foresight. Under the more palatable assumption that cyclical variations are not perfectly foreseen, the tax smoothing approach can have trouble explaining cyclical fiscal policy in the long run. Specifically, in environments with incomplete markets, the approach can imply that the government should self-insure, eventually accumulating sufficient assets to finance government spending out of the interest earnings from these assets (Aiyagari et al (2002)).[8] In this case, the model predicts no long run cyclical pattern in debt,

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[8] Different conclusions arise when there are complete markets and the government can issue state-contingent debt. We focus on the incomplete markets assumption here because we feel that it is the most appropriate for

taxes, or public spending. In a world in which spending shocks drive fiscal policy, Aiyagari et al (2002) show that the tax smoothing model can generate plausible fiscal dynamics if the government faces an upper bound on how many assets it can accumulate, but it is not clear why such a bound should exist.

In the model of this paper, a social planner would choose to gradually accumulate assets so that in the long run all fiscal policies would be constant. However, as in Battaglini and Coate (2008), incorporating political decision-making resolves this difficulty. Legislators ability to redistribute tax revenues back to their districts creates an endogenous bound on how many assets the government will accumulate. In this sense, the theory can be thought of as naturally extending the work of Aiyagari et al (2002).

This paper is distinctive in assessing the quantitative implications of its theory. Previous work has been limited to developing and assessing the qualitative implications of different theories. Thus, the predictions concerning the nature of empirical correlations between fiscal policies and output have been derived and tested, but the ability of different theories to explain the size of correlations has not been assessed. Aiyagari et al (2002) simulate their model to study the dynamic behavior of debt in examples but the model is not calibrated and, in any case, the driver of fiscal policy is spending shocks rather than the business cycle.

Our quantitative analysis complements the recent work of Azzimonti, Battaglini, and Coate (2009) who calibrate the Battaglini and Coate (2008) model. The focus of the Azzimonti et al paper is to provide a quantitative assessment of the case for imposing a balanced budget rule in the Battaglini and Coate model. Since the driver of fiscal policy is independent and identically distributed spending shocks, Azzimonti et al calibrate their model by matching moments of peace-time and wartime spending, as well as moments of the debt distribution. By contrast, the purpose of this paper is to explore the cyclical behavior of fiscal policies and the driver of fiscal policy is persistent productivity shocks. Thus, the key to our calibration exercise is matching the cyclical properties of GDP.

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a positive analysis. We refer the reader to Chari, Christiano and Kehoe (1994) for a comprehensive analysis of optimal fiscal policy in a real business cycle model with complete markets and to Marcet and Scott (2009) for an interesting effort to empirically test between the complete and incomplete market assumptions.

3 The model

3.1 The economic environment

A continuum of infinitely-lived citizens live in n identical districts indexed by $i = 1, ..., n$. The size of the population in each district is normalized to be one. There is a single (nonstorable) consumption good, denoted by z, that is produced using a single factor, labor, denoted by l, with the linear technology $z = wl$. There is also a public good, denoted by g, that can be produced from the consumption good according to the linear technology $g = z/p$.

Citizens consume the consumption good, benefit from the public good, and supply labor. Each citizen's per period utility function is

$$z + A\frac{g^{1-\sigma}}{1-\sigma} - \frac{l^{(1+1/\varepsilon)}}{\varepsilon+1}, \tag{1}$$

where $\sigma > 0$ and $\varepsilon > 0$.[9] The parameter A measures the utility from the public good relative to the utility from consumption and the parameter σ controls the elasticity of the citizens' utility with respect to the public good. Citizens discount future per period utilities at rate β.

The productivity of labor w varies across periods in a random way, reflecting the business cycle.[10] Specifically, the economy can either be in a *boom* or a *recession*. Labor productivity is w_H in a boom and w_L in a recession, where $w_L < w_H$. The state of the economy follows a first order Markov process, with transition matrix

$$\begin{bmatrix} \alpha_{LL} & \alpha_{LH} \\ \alpha_{HL} & \alpha_{HH} \end{bmatrix}.$$

Thus, conditional on the economy being in a recession, the probability of remaining in a recession is α_{LL} and the probability of transitioning to a boom is α_{LH}. Similarly, conditional on being in a boom, the probability of remaining in a boom is α_{HH} and the probability of transitioning to a recession is α_{HL}. Though in many environments it is natural to assume that states are persistent, this assumption is not necessary for our results. However, we do require that α_{HH} exceeds α_{LH}, so that the economy is more likely to be in a boom if it was in a boom the previous period.[11]

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[9] When $\sigma = 1$, the utility from the public good becomes $A\log(g)$.

[10] In Battaglini and Coate (2008), productivity is constant and the value of the public good A is random.

[11] Our basic model assumes that in the "up-part" of the business cycle there is a single productivity level w_H, and in the "down-part" a single productivity level w_L. Thus, within booms and recessions, there is no variation

There is a competitive labor market and competitive production of the public good. Thus, the wage rate is equal to w_H in a boom and w_L in a recession and the price of the public good is p. There is also a market in risk-free one period bonds. The assumption of a constant marginal utility of consumption implies that the equilibrium interest rate on these bonds must be $\rho = 1/\beta - 1$. At this interest rate, citizens will be indifferent as to their allocation of consumption across time.

3.2 Government policies

The public good is provided by the government. The government can raise revenue by levying a proportional tax on labor income. It can also borrow and lend by selling and buying bonds. Revenues can not only be used to finance the provision of the public good but can also be diverted to finance targeted district-specific transfers which are interpreted as (non-distortionary) pork-barrel spending.

Government policy in any period is described by an $n+3$-tuple $\{\tau, g, x, s_1, ..., s_n\}$, where τ is the income tax rate, g is the amount of the public good provided, x is the amount of bonds sold, and s_i is the proposed transfer to district i's residents. When x is negative, the government is buying bonds. In each period, the government must also repay any bonds that it sold in the previous period. Thus, if it sold b bonds in the previous period, it must repay $(1+\rho)b$ in the current period. The government's initial debt level in period 1 is given exogenously and is denoted by b_0.

In a period in which government policy is $\{\tau, g, x, s_1, ..., s_n\}$ and the state of the economy (i.e., boom or recession) is $\theta \in \{L, H\}$, each citizen will supply an amount of labor

$$l_\theta^*(\tau) = \arg\max_l \{w_\theta(1-\tau)l - \frac{l^{(1+1/\varepsilon)}}{\varepsilon+1}\}. \tag{2}$$

It is straightforward to show that $l_\theta^*(\tau) = (\varepsilon w_\theta(1-\tau))^\varepsilon$, so that ε is the elasticity of labor supply. A citizen in district i who simply consumes his net of tax earnings and his transfer will obtain a per period utility of $u_\theta(\tau, g) + s_i$, where

$$u_\theta(\tau, g) = \frac{\varepsilon^\varepsilon (w_\theta(1-\tau))^{\varepsilon+1}}{\varepsilon+1} + A\frac{g^{1-\sigma}}{1-\sigma}. \tag{3}$$

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in productivity. While this is a rather spartan conception of a business cycle, the model can be extended to incorporate within state productivity shocks by assuming that productivity in state θ is given by $w_\theta + \omega$ where ω is an i.i.d "shock" with mean zero, range $[-\overline{\omega}, \overline{\omega}]$. Though the introduction of i.i.d shocks makes the distinction between booms and recessions less clear-cut, the equilibrium of the extended model has the same structure as the equilibrium of the simpler model described in the text and produces the same predictions of the key correlation between macro variables. A more complete analysis of this extension is available from the authors.

Since citizens are indifferent as to their allocation of consumption across time, their lifetime expected utility will equal the value of their initial bond holdings plus the payoff they would obtain if they simply consumed their net earnings and transfers in each period.

Government policies must satisfy three feasibility constraints. The first is that revenues must be sufficient to cover expenditures. To see what this implies, consider a period in which the initial level of government debt is b, the policy choice is $\{\tau, g, x, s_1, ..., s_n\}$, and the state of the economy is θ. Expenditure on public goods and debt repayment is $pg + (1+\rho)b$, tax revenue is

$$R_\theta(\tau) = n\tau w_\theta l^*_\theta(\tau) = n\tau w_\theta(\varepsilon w_\theta(1-\tau))^\varepsilon, \tag{4}$$

and revenue from bond sales is x. Letting the *net of transfer surplus* (i.e., the difference between revenues and spending on public goods and debt repayment) be denoted by

$$B_\theta(\tau, g, x; b) = R_\theta(\tau) - pg + x - (1+\rho)b, \tag{5}$$

the constraint requires that $B_\theta(\tau, g, x; b) \geq \sum_i s_i$.

The second constraint is that the district-specific transfers must be non-negative (i.e., $s_i \geq 0$ for all i). This rules out financing public spending via district-specific lump sum taxes. With lump sum taxes, there would be no need to impose the distortionary labor tax and hence no tax smoothing problem.

The third and final constraint is that the amount of government borrowing must be feasible. In particular, there is an upper limit $\overline{x}$ on the amount of bonds the government can sell. This limit is motivated by the unwillingness of borrowers to hold bonds that they know will not be repaid. If the government were borrowing an amount x such that the interest payments exceeded the maximum possible tax revenues in a recession; i.e., $\rho x > \max_\tau R_L(\tau)$, then, if the economy were in recession, it would be unable to repay the debt *even if it provided no public goods or transfers.* Thus, the maximum level of debt is $\overline{x} = \max_\tau R_L(\tau)/\rho$.

We avoid assuming that there is any "ad hoc" limit on the amount of bonds that the government can purchase (see Aiyagari et al (2002)). In particular, the government is allowed to hold sufficient bonds to permit it to always finance the Samuelson level of the public good from the interest earnings. This level of bonds is given by $\underline{x} = -pg_S/\rho$, where g_S is the level of the public good that satisfies the *Samuelson Rule.*[12] Since the government will never want to hold more bonds

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[12] The Samuelson Rule is that the sum of marginal benefits equal the marginal cost, which means that g_S satisfies the first order condition that $nAg^{-\sigma} = p$.

than this, there is no loss of generality in constraining the choice of debt to the interval $[\underline{x}, \overline{x}]$ and we will do this below.[13] We also assume that the initial level of government debt, b_0, belongs to the interval $(\underline{x}, \overline{x})$.

3.3 The political process

Government policy decisions are made by a legislature consisting of representatives from each of the n districts. One citizen from each district is selected to be that district's representative. Since all citizens have the same policy preferences, the identity of the representative is immaterial and hence the selection process can be ignored.[14] The legislature meets at the beginning of each period. These meetings take only an insignificant amount of time, and representatives undertake private sector work in the rest of the period just like everybody else. The affirmative votes of $q < n$ representatives are required to enact any legislation.

To describe how legislative decision-making works, suppose the legislature is meeting at the beginning of a period in which the current level of public debt is b and the state of the economy is θ. One of the legislators is randomly selected to make the first proposal, with each representative having an equal chance of being recognized. A proposal is a policy $\{\tau, g, x, s_1, ..., s_n\}$ that satisfies the feasibility constraints. If the first proposal is accepted by q legislators, then it is implemented and the legislature adjourns until the beginning of the next period. At that time, the legislature meets again with the difference being that the initial level of public debt is x and that the state of the economy may have changed. If, on the other hand, the first proposal is not accepted, another legislator is chosen to make a proposal. There are $T \geq 2$ such proposal rounds, each of which takes a negligible amount of time. If the process continues until proposal round T, and the proposal made at that stage is rejected, then a legislator is appointed to choose a default policy. The only restrictions on the choice of a default policy are that it be feasible and that it involve a uniform district-specific transfer (i.e., $s_i = s_j$ for all i, j).

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[13] By assuming that the government can choose to borrow any amount in the interval $[\underline{x}, \overline{x}]$, we are implicitly assuming that labor productivity is sufficiently high that the amount spent on public goods is never higher than national income. A sufficient condition for this is that $nw_L(\varepsilon w_L(\frac{\varepsilon}{1+\varepsilon}))^{\varepsilon} > pg_S$ (see Battaglini and Coate (2008) for details).

[14] While citizens may differ in their bond holdings, this has no impact on their policy preferences.

4 The social planner's solution

To create a normative benchmark with which to compare the political equilibrium, we begin by describing what fiscal policy would look like if policies were chosen by a social planner who wished to maximize aggregate utility. The planner's problem can be formulated recursively.[15] In a period in which the current level of public debt is b and the state of the economy is θ, the problem is to choose a policy $\{\tau, g, x, s_1, ..., s_n\}$ to solve:

$$\begin{array}{c} \max\ u_\theta(\tau, g) + \frac{\sum_i s_i}{n} + \beta[\alpha_{\theta H} v_H^\circ(x) + \alpha_{\theta L} v_L^\circ(x)] \\ s.t.\ \ s_i \geq 0 \ \text{ for all } i,\ \sum_i s_i \leq B_\theta(\tau, g, x; b),\ \&\ x \in [\underline{x}, \overline{x}], \end{array} \tag{6}$$

where $v_\theta^\circ(x)$ denotes the representative citizen's value function in state θ (net of bond holdings).

Surplus revenues will optimally be rebated back to citizens and hence $\sum_i s_i = B_\theta(\tau, g, x; b)$. Thus, we can reformulate the problem as choosing a tax-public good-debt triple (τ, g, x) to solve:

$$\begin{array}{c} \max\ u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta H} v_H^\circ(x) + \alpha_{\theta L} v_L^\circ(x)] \\ s.t.\ \ B_\theta(\tau, g, x; b) \geq 0\ \&\ x \in [\underline{x}, \overline{x}]. \end{array} \tag{7}$$

The problem in this form is fairly standard. The citizen's value functions v_L° and v_H° solve the functional equations

$$v_\theta^\circ(b) = \max_{(\tau,g,x)} \left\{ \begin{array}{c} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta H} v_H^\circ(x) + \alpha_{\theta L} v_L^\circ(x)] \\ s.t.\ \ B_\theta(\tau, g, x; b) \geq 0\ \&\ x \in [\underline{x}, \overline{x}] \end{array} \right\} \ \theta \in \{L, H\} \tag{8}$$

and the planner's policies in state θ, $\{\tau_\theta^\circ(b),\ g_\theta^\circ(b),\ x(b)\}$, are the optimal policy functions for this program.

In any given state (b, θ) the planner's optimal policies $\{\tau_\theta^\circ(b),\ g_\theta^\circ(b),\ x_\theta^\circ(b)\}$ are implicitly defined by three conditions. The first is that the social marginal benefit of the public good is equal to the social marginal cost of financing it; that is,

$$nAg^{-\sigma} = p(\frac{1-\tau}{1-\tau(1+\varepsilon)}). \tag{9}$$

To interpret this, note that $(1-\tau)/(1-\tau(1+\varepsilon))$ measures the *marginal cost of public funds* (MCPF) - the social cost of raising an additional unit of revenue via a tax increase. The term on

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[15] Because the interest rate is constant, there is no time inconsistency problem in this model. Thus, assuming that the planner chooses policies period-by-period yields the same results as assuming that he is a *Ramsey planner*, choosing a time path of policies at the beginning of period 1.

the right hand side therefore represents the cost of financing an additional unit of the public good. The condition is just the *Samuelson Rule* modified to take account of the fact that taxation is distortionary and it determines the optimal public good level for any given tax rate. The second condition is that the marginal cost of public funds today equals the expected marginal cost of debt tomorrow; that is,[16]

$$\frac{1-\tau}{1-\tau\,(1+\varepsilon)} = -n\beta[\alpha_{\theta H} v_H^{\circ\prime}(x) + \alpha_{\theta L} v_L^{\circ\prime}(x)]. \tag{10}$$

This ensures that, on the margin, the cost of financing public goods via taxation equals that of financing them by issuing debt. The final condition is that the net of transfer surplus be zero; that is,

$$B_\theta(\tau, g, x; b) = 0. \tag{11}$$

This implies that the planner raises no more revenues than are necessary to finance public good spending.

Using these conditions, it is possible to show that for each state θ the optimal tax rate and debt level are increasing in b and the optimal public good level is decreasing in b. Using the *Envelope Theorem*, it is also straightforward to show that the marginal cost of debt tomorrow in state θ is just the marginal cost of public funds tomorrow in state θ; that is,

$$-n\beta v_\theta^{\circ\prime}(x) = \left(\frac{1-\tau_\theta^\circ(x)}{1-\tau_\theta^\circ(x)\,(1+\varepsilon)}\right). \tag{12}$$

Substituting this into (10), yields the Euler equation for the planner's problem:

$$\frac{1-\tau_\theta^\circ(b)}{1-\tau_\theta^\circ(b)\,(1+\varepsilon)} = \alpha_{\theta H}\left(\frac{1-\tau_H^\circ(x(b))}{1-\tau_H^\circ(x(b))\,(1+\varepsilon)}\right) + \alpha_{\theta L}\left(\frac{1-\tau_L^\circ(x(b))}{1-\tau_L^\circ(x(b))\,(1+\varepsilon)}\right). \tag{13}$$

This equation tells us that the optimal debt level equalizes the current MCPF with the corresponding expected MCPF and implies that *the MCPF obeys a martingale.*[17] The condition illustrates the planner's desire to smooth taxation between periods.

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[16] Note that in deriving (10) we are ignoring the upperbound $x \leq \overline{x}$. We show in the Appendix (Section 10.6) that this is without loss of generality.

[17] Bohn (1990) establishes this result for a stochastic version of the tax smoothing model studied by Barro (1979). Aiyagari et al (2002) show a similar result for the planner's solution in a model very similar to ours. To ease the comparison, however, note that the negative of their Lagrangian multiplier ψ_t corresponds to our MCPF minus one. It should also be noted that in their model the planner's MCPF follows a supermartingale because the upper bound on debt will bind with positive probability. This however depends on the fact that g_t is an exogenous process. This can not happen in our framework because g_t is endogenous.

The Euler equation (13) is the key to understanding the dynamic evolution of the system. It implies that the planner raises debt in a recession and lowers it in a boom. He raises debt in a recession because he anticipates that the economic environment can only improve in the future. If it does improve, the MCPF will be lower since tax rates are lower in booms than in recessions.[18] Thus, debt must increase to maintain equation (13). Likewise, when the economy is in a boom, the planner anticipates that the economic environment can only get worse in the future and thus decreases debt. The upshot is that debt behaves counter-cyclically. On the other hand, public good spending behaves pro-cyclically with spending increasing in booms and falling in recessions.

What happens in the long run? Since the MCPF is a convex function of the tax rate τ, the martingale property implies that the current tax rate exceeds the expected tax rate. Thus, the tax rate behaves as a supermartingale.[19] The *Martingale Convergence Theorem* therefore implies that the tax rate converges to a constant with probability one. The only steady state compatible with a constant tax rate, is a steady state in which the government has accumulated such a large pool of assets that spending needs can be financed out of the interest earned, and taxation is zero. Indeed, if this were not true (and taxation were positive), then the tax rate would have to depend on θ. We can therefore conclude that the social planner's solution converges to a steady state in which the debt level is $\underline{x}$, the tax rate is 0, and the public good level is g_S.[20]

The key take away point is that, while in the short run debt displays the counter-cyclical pattern usually associated with the tax smoothing approach, this disappears in the long run. Moreover, all other fiscal policy variables are also constant. This observation underscores the point made in Section 2: when cyclical variations are not perfectly anticipated, the tax smoothing approach has difficulty explaining cyclical fiscal policy in the long run.

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[18] While tax rates being lower in booms than in recessions (i.e., $r_H^{\circ}(b) < r_L^{\circ}(b)$) seems natural, it may not be immediate how to prove it. Since the planner's solution is a special case of the political equilibrium when $q = n$, the result will follow from Lemma 2 in Section 7.

[19] If the MCPF is linear in the tax rate, as assumed in Bohn (1990), the tax rate behaves as a martingale as was conjectured by Barro (1979).

[20] A similar conclusion holds when public spending shocks rather than revenue shocks are the driver of fiscal policy (see Aiyagari et al (2002) and Battaglini and Coate (2008)). However, with public spending shocks, optimal public good spending is uncertain and the government accumulates sufficient assets to finance the highest level of such spending. Interest earnings in excess of optimal public good spending are rebated back to the citizens via a uniform transfer. In this model, the planner does not need to use transfers since optimal public good spending is constant.

5 The political equilibrium

Following Battaglini and Coate (2008), we look for a symmetric Markov-perfect equilibrium in which any representative selected to propose at round $r \in \{1, ..., T\}$ of the meeting at some time t makes the same proposal and this depends only on the current level of public debt (b) and the state of the economy (θ). As standard in the theory of legislative voting, we assume that legislators vote for a proposal if they prefer it (weakly) to continuing on to the next proposal round. We focus, without loss of generality, on equilibria in which at each round r, proposals are immediately accepted by at least q legislators, so that on the equilibrium path, no meeting lasts more than one proposal round. Accordingly, the policies that are actually implemented in equilibrium are those proposed in the first round.

Let $\{\tau_\theta(b), g_\theta(b), x(b)\}$ denote the tax rate, public good and public debt policies that are implemented in equilibrium when the state is (b, θ). In addition, let $v_\theta(b)$ denote the common legislator's value function when the state of the economy is θ. Reflecting the fact that legislators are ex ante equally likely to receive transfers, this is defined recursively by:

$$v_\theta(b) = u_\theta(\tau_\theta(b), g_\theta(b)) + \frac{B_\theta(\tau_\theta(b), g_\theta(b), x(b); b)}{n} + \beta[\alpha_{\theta H} v_H(x(b)) + \alpha_{\theta L} v_L(x(b))]. \tag{14}$$

This is also the value function for each citizen, since representatives obtain the same payoffs as their constituents. We say that an equilibrium is *well-behaved* if the associated legislators' value functions v_L and v_H are continuous and concave on $[\underline{x}, \overline{x}]$. In the Appendix we prove:

Proposition 1. *There exists a well-behaved equilibrium.*

Henceforth, when we refer to an "equilibrium", it is to be understood that it is well-behaved.

5.1 Equilibrium policies

To understand equilibrium behavior, note that to get support for his proposal the proposer must obtain the votes of $q - 1$ other representatives. Accordingly, given that utility is transferable, he is effectively making decisions to maximize the utility of q legislators. It is therefore *as if* a randomly chosen *minimum winning coalition* (mwc) of q representatives is selected in each period and this coalition chooses a policy choice to maximize its aggregate utility. Formally, this means that, when the state is (b,θ), the tax-public good-debt triple (τ, g, x) solves the problem:

$$\max_{\tau,g,x}\left\{\begin{array}{c} u_\theta(\tau,g)+\frac{B_\theta(\tau,g,x;b)}{q}+\beta[\alpha_{\theta H}v_H(x)+\alpha_{\theta L}v_L(x)] \\ \\ s.t.\quad B_\theta(\tau,g,x;b)\geq 0\ \&\ x\in[\underline{x},\overline{x}]\,. \end{array}\right\} \tag{15}$$

In any given state (b,θ), there are two possibilities: either the mwc will provide pork to the districts of its members or it will not. Providing pork requires reducing public good spending or increasing taxation in the present or the future (if financed by issuing additional debt). When b is high and/or the economy is in a recession, the opportunity cost of revenues may be too high to make this attractive. In this case, the mwc will not provide pork, so $B_\theta(\tau,g,x;b)=0$. From (15), it is clear that the outcome will then be *as if* the mwc is maximizing the utility of the legislature as a whole. Indeed, the policy choice will be identical to that a benevolent planner would choose in the same state and with the same value function.

When b is low and/or the economy is in a boom, the opportunity cost of revenues is lower. Less tax revenues need to be devoted to debt repayment when b is low and both current and expected future tax revenues are more plentiful when the economy is in a boom. As a result, the mwc will allocate revenues to pork and policies will diverge from those that would be chosen by a planner. Interestingly, it turns out that this diversion of resources toward pork, effectively creates lower bounds on how low the tax rate and debt level can go, and an upper bound on how high the level of the public good can be.

To show this, we must first characterize the policy choices that the mwc selects when it provides pork. Consider again problem (15) and suppose that the constraint $B_\theta(\tau,g,x;b)\geq 0$ is not binding. Using the first-order conditions for this problem, we find that the optimal tax rate τ^* satisfies the condition that

$$\frac{1}{q}=\frac{\left[\frac{1-\tau^*}{1-\tau^*(1+\varepsilon)}\right]}{n}. \tag{16}$$

The condition says that the benefit of raising taxes in terms of increasing the per-coalition member transfer $(1/q)$ must equal the per-capita MCPF. Similarly, the optimal public good level g^* satisfies the condition that

$$Ag^{*-\sigma}=\frac{p}{q}. \tag{17}$$

This says that the per-capita benefit of increasing the public good must equal the per-coalition member reduction in transfers that providing the additional unit necessitates. The optimal public debt level x_θ^* satisfies the condition that

$$x_\theta^* = \arg\max\{\frac{x}{q} + \beta[\alpha_{\theta H}v_H(x) + \alpha_{\theta L}v_L(x)] : x \in [\underline{x}, \overline{x}]\}. \tag{18}$$

The optimal level balances the benefit of increasing debt in terms of increasing the per-coalition member transfer with the expected per-capita cost of an increase in the debt level.

We can now make precise how the legislature's ability to divert resources toward pork-barrel spending effectively creates endogenous bounds on the policy choices.

Proposition 2. *The equilibrium value functions $v_H(b)$ and $v_L(b)$ solve the system of functional equations*

$$v_\theta(b) = \max_{(\tau,g,x)} \left\{ \begin{array}{c} u_\theta(\tau,g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta L}v_L(x) + \alpha_{\theta H}v_H(x)] \\ \\ s.t. \;\; B_\theta(\tau,g,x;b) \geq 0,\ \tau \geq \tau^*,\ g \leq g^* \ \&\ x \in [x_\theta^*, \overline{x}] \end{array} \right\} \quad \theta \in \{L, H\} \tag{19}$$

and the equilibrium policies $\{\tau_\theta(b),\ g_\theta(b),\ x(b)\}$ are the optimal policy functions for this program.

Thus, the equilibrium policy choices solve a *constrained planner's problem* in which the tax rate can not fall below τ^*, the public good level can not exceed g^*, and debt can not fall below the state contingent threshold x_θ^*.[21] However, there is a fundamental difference with the planner's problem (8). The thresholds that constrain the policies are endogenous because they depend on the economic fundamentals and, in the case of x_L^* and x_H^*, on the equilibrium: so rather than being constraints that *affect* the value function, they are determined simultaneously *with* the value function.

Given Proposition 2, the nature of the equilibrium policies in a given state θ is clear. For any equilibrium, define b_θ^* to be the value of debt such that the triple $(\tau^*, g^*, x_\theta^*)$ satisfies the constraint that $B_\theta(\tau^*, g^*, x_\theta^*; b) = 0$. This is given by:

$$b_\theta^* = \frac{\tau_\theta(\tau^*) + x_\theta^* - pg^*}{1+\rho}. \tag{20}$$

Then, if the debt level b is such that $b \leq b_\theta^*$ the tax-public good-debt triple is $(\tau^*, g^*, x_\theta^*)$ and the net of transfer surplus $B_\theta(\tau^*, g^*, x_\theta^*; b)$ is used to finance transfers.[22] If $b > b_\theta^*$ the budget

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[21] This result extends Proposition 4 of Battaglini and Coate (2008) by showing that when shocks are persistent the lower bound on debt in the constrained planning problem will be state-contingent.

[22] Recall that in the planner's solution, the government never makes transfers. These transfers, therefore, are a political distortion. Intuitively, citizens could be made better off ex ante by reducing transfers and decreasing the tax rate.

constraint binds so that no transfers are given. The tax rate and public debt level strictly exceed (τ^*, x_θ^*) and the public good level is strictly less than g^*. In this case, therefore, the solution can be characterized by obtaining the first order conditions for problem (19) with only the budget constraint binding. These are conditions (9), (10), and (11) except with the equilibrium value functions. It is easy to show that the tax rate and debt level are increasing in b, while the public good level is decreasing in b.[23]

To compare policies across states, the key step is to understand how the political constraints change over the cycle, i.e. the relationship between x_L^* and x_H^*. To characterize these debt levels we use (18) and the following Lemma:

Lemma 1. *For each state of the economy $\theta \in \{L, H\}$, the equilibrium value function $v_\theta(\cdot)$ is differentiable for all b such that $b \neq b_\theta^*$. Moreover:*

$$-v_\theta'(b) = \begin{cases} \left(\frac{1-\tau_\theta(b)}{1-\tau_\theta(b)(1+\varepsilon)}\right)\left(\frac{1+\rho}{n}\right) & \text{if } \ b > b_\theta^* \\ \left(\frac{1+\rho}{n}\right) & \text{if } \ b < b_\theta^* \end{cases} . \tag{21}$$

To understand this result, recall that when the initial debt level b exceeds b_θ^*, there is no pork, so to pay back an additional unit of debt requires an increase in taxes. This means that the cost of an additional unit of debt is equal to the repayment amount $1+\rho$ multiplied by the per capita MCPF. By contrast, when b is less than b_θ^*, pork will be reduced to pay back additional debt since that is the marginal use of resources. The cost of an additional unit of debt is thus equal to $1+\rho$ multiplied by the expected per capita reduction in pork which is $1/n$. Notice that the value function is not differentiable at $b = b_\theta^*$. The left hand derivative at $b = b_\theta^*$ is equal to $(1+\rho)/n$ and the right hand derivative is equal to $(1+\rho)/q$ (since the tax rate $\tau_\theta(x)$ equals τ^* at $b = b_\theta^*$).[24] This discontinuity reflects the fact that increasing taxes is more costly than reducing pork because the marginal cost of taxation exceeds 1.

Using Lemma 1 and the first order conditions for problem (18), we can now show that:

Lemma 2. *In any equilibrium: $b_L^* < x_L^* \leq x_H^* \leq b_H^*$.*

The proof of this result also implies that if $x_L^* < b_H^*$, then $x_L^* < x_H^*$. Thus, the only circumstance in which $x_L^* = x_H^*$ is when both equal b_H^*. While this is possible, it only arises when α_{LH}

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[23] Details are available from the authors upon request.

[24] The set of sub-gradients of the value function v_θ at $x = b_\theta^*$ is $[-(\frac{1+\rho}{q}), -(\frac{1+\rho}{n})]$.

is sufficiently close to α_{HH} to make a recession barely persistent. Under these circumstances, legislators would not find it optimal to borrow less when providing pork during a recession than during a boom because the recession is sufficiently likely to revert to a boom. From here on, we will assume that the transition probabilities are such that $x_L^* < x_H^*$ which we see as the most interesting case.[25]

With Lemma 2 in hand, we can provide a complete picture of how fiscal policy changes with the state of the economy θ for any level of debt b. When b is less than b_L^* the mwc provides pork in both booms and recessions (since $b_L^* < b_H^*$). In this case, the tax rate and public good provision are constant across states, respectively at τ^* and g^*, while debt will be higher in a boom than in a recession (respectively, x_L^* versus x_H^*). Tax revenues will be higher in a boom and these extra revenues, together with the extra borrowing, will be used to finance higher levels of pork-barrel spending. When b is between b_L^* and b_H^* the mwc provides pork in a boom but not in a recession. In this case, taxes will be higher in a recession and public good provision will be lower. Over this interval of initial debt levels, the new level of debt will be constant in a boom, but increasing in a recession. We show in the Appendix that there will be a threshold debt level $\widehat{b} \in (b_L^*, b_H^*)$ such that new debt will be higher in a recession if and only if $b > \widehat{b}$. Finally, when b exceeds b_H^* the mwc does not provide pork in either state. In this range, public good levels will be lower in a recession ($g_L(b) < g_H(b)$), tax rates will be higher ($\tau_L(b) > \tau_H(b)$), and public borrowing will be higher ($x_L(b) > x_H(b)$).

5.2 Policy dynamics

Having understood the nature of equilibrium policies within and across states, we are now ready to explore their behavior over the business cycle. The key policy to understand is public debt, since the cyclical behavior of all the remaining fiscal policies will follow from the behavior of debt given the results we already have. The next result characterizes debt policies in booms and recessions:

Lemma 3. *In any equilibrium: (i)* $x_L(b) > b$ *for all* $b \in [\underline{x}, \overline{x})$*, and, (ii)* $x_H(b) > b$ *for all* $b \in (\underline{x}, x_H^*)$ *and* $x_H(b) < b$ *for all* $b \in (x_H^*, \overline{x}]$*.*

Part (i) implies that the debt level always increases in a recession. Intuitively, if we are in a recession today, the economic environment can only improve in the future. This makes it

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[25] A sufficient condition for this to be true is that recessions are sufficiently persistent, that is α_{LL} is sufficiently high.

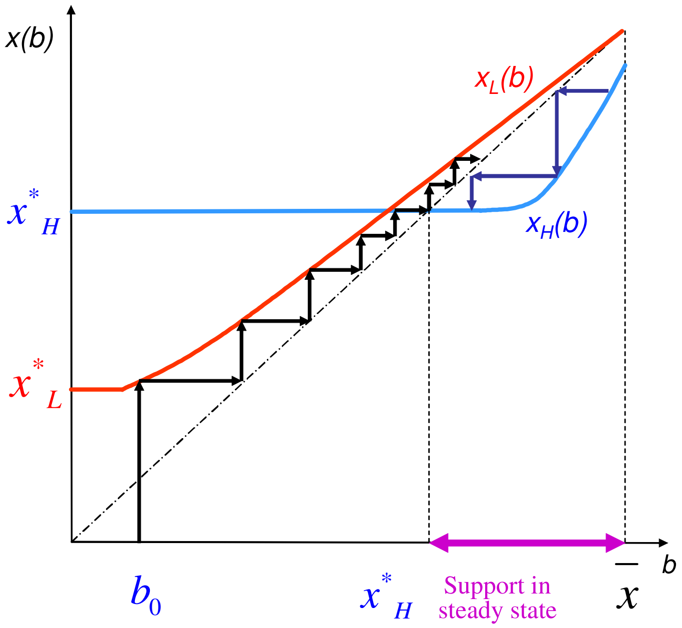

Figure 1: Equilibrium dynamics

worthwhile for the legislature to increase debt. Part (ii) implies that the debt level decreases in a boom if the initial debt level exceeds x and increases otherwise. Figure 1 graphs the functions $x_L(b)$ and $x_H(b)$.

The cyclical behavior of debt can be easily inferred from Lemma 3. The first time the economy moves from recession to boom it is possible for debt to behave pro-cyclically - jumping up when the economy enters the boom. This occurs if the level of accumulated debt is less than x^*_H when the first boom arrives. The boom increases both current and expected future productivity, which reduces the expected marginal cost of debt. Debt jumps to a level at which equality between the marginal benefit of pork and the expected marginal cost of borrowing is reestablished and, during this process, a "pork-fest" occurs. This is very similar to the logic underlying Lane and Tornell's voracity effect.

After this initial transition, however, debt must behave counter-cyclically.[26] For once such a pro-cyclical debt expansion has occurred it can never happen again. This is clear from Figure 1. The debt level is bounded below by x^*_H in a boom and, as demonstrated in Lemma 3, it is

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[26] As noted earlier, the voracity effect papers just consider the implications of a one time positive income shock.

increasing in a recession. Once the first boom has occurred and debt has jumped up to x_H^*, debt will increase when the economy goes into recession. When another boom occurs debt will decrease down to x_H^* and then remain constant. Moreover, we can show that no matter what the economy's initial debt level, the same distribution of debt emerges in the long run. To summarize:

Proposition 3. *In any equilibrium, the debt distribution strongly converges to a unique, non degenerate, invariant distribution with support on* $[x_H^*, \overline{x}]$. *The dynamic pattern of debt is countercyclical. When the economy enters a recession, debt will increase and will continue to increase as long as the recession persists. When the economy enters a boom, debt decreases and, during the boom, continues to decline until it reaches* x_H^*.

This Proposition implies that debt and GDP should be negatively correlated. Debt levels go down upon entering a boom and continue to decline over the course of a boom. By contrast, debt levels are increasing over the course of a recession. Since GDP levels are increasing over the course of a boom and decreasing over the course of a recession, debt and GDP are always moving in the opposite direction.[27]

Since the remaining fiscal policies are all functions of debt, Proposition 3 implies that the distribution of these policies will also be invariant in the long term. Combining Proposition 3 with our understanding of equilibrium policies from the previous section, allows us to predict their long-run cyclical behavior.

Proposition 4. *In any equilibrium, in the long run, when the economy enters a recession, the tax rate increases and public good provision decreases. Moreover, the tax rate will continue to increase and public good provision will continue to decrease as long as the recession persists. When the economy enters a boom, the tax rate decreases and public good provision increases. During the boom, the tax rate continues to decline and public good provision continues to increase until they reach, respectively,* τ^* *and* g^*. *Pork-barrel spending will not occur during a recession. Moreover, it will only occur during a boom once the debt accumulated during prior recessions has been paid off and debt has reached* x_H^*.

The intuition behind this proposition is straightforward. When a recession arrives, both current and expected productivity are reduced. The government reacts to this by tightening fiscal

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[27] While productivity levels are constant, GDP levels are increasing (decreasing) during booms (recessions) because tax rates are decreasing (increasing) (Proposition 4).

policy: reducing public good expenditure, and increasing taxes and debt. After the first period of recession, debt is higher than before. Thus, if the economy remains in recession, public good spending will further decrease, and taxes and debt will further increase. This process stops at the arrival of a boom. The increase in both current and expected productivity allows the government to reduce taxes and debt, and increase public good spending. If the economy remains in a boom, the mechanism just described is reversed: the reduction in debt implies that taxes and debt further decrease, and public good spending further increases.

Proposition 4 implies that the tax rate is negatively correlated with GDP. It also implies that public spending is positively correlated with GDP.[28] The equilibrium changes in public spending and taxes therefore serve to amplify the business cycle. These predictions are distinctive and serve to nicely differentiate the predictions of our neoclassical theory from what would be expected if government were following a Keynesian counter-cyclical fiscal policy. For, in a recession, a Keynesian government would reduce taxes and increase public spending to bolster aggregate demand.

There is one more implication of the theory concerning the dynamic evolution of fiscal variables worthy of note. This concerns the MCPF. As discussed in Section 4, the social planner smooths taxation over time by equalizing the current MCPF with the expected MCPF next period, implying that the MCPF behaves as a martingale. In a political equilibrium, whether the mwc is providing pork or not, the debt level must be such that the MCPF today equals the expected marginal cost of debt tomorrow; that is,[29]

$$\frac{1-\tau_\theta(b)}{1-\tau_\theta(b)\,(1+\varepsilon)} = -n\beta[\alpha_{\theta H} v_H'(x_\theta(b)) + \alpha_{\theta L} v_L'(x_\theta(b))]. \tag{22}$$

If, for example, the MCPF exceeded the expected marginal cost of debt, the mwc could shift the financing of its spending program from taxation to debt and make each coalition member better off. Combining this equation with Lemma 1, we immediately obtain:

Proposition 5. *In any equilibrium, the marginal cost of public funds is a submartingale; that is,*

$$\frac{1-\tau_\theta(b)}{1-\tau_\theta(b)\,(1+\varepsilon)} \leq \alpha_{\theta H}[\frac{1-\tau_H(x(b))}{1-\tau_H(x(b))\,(1+\varepsilon)}] + \alpha_{\theta L}[\frac{1-\tau_L(x(b))}{1-\tau_L(x(b))\,(1+\varepsilon)}], \tag{23}$$

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[28] Notice, however, that the theory provides no predictions on the correlation between spending *as a proportion of* GDP and GDP. This is because when GDP increases both the numerator and the denominator of the ratio increase and which increases more will depend on how the parameters of the model are calibrated.

[29] In deriving (22) we are ignoring the upperbound $x \leq \overline{x}$. The proof of Proposition 5 establishes that this is without loss of generality.

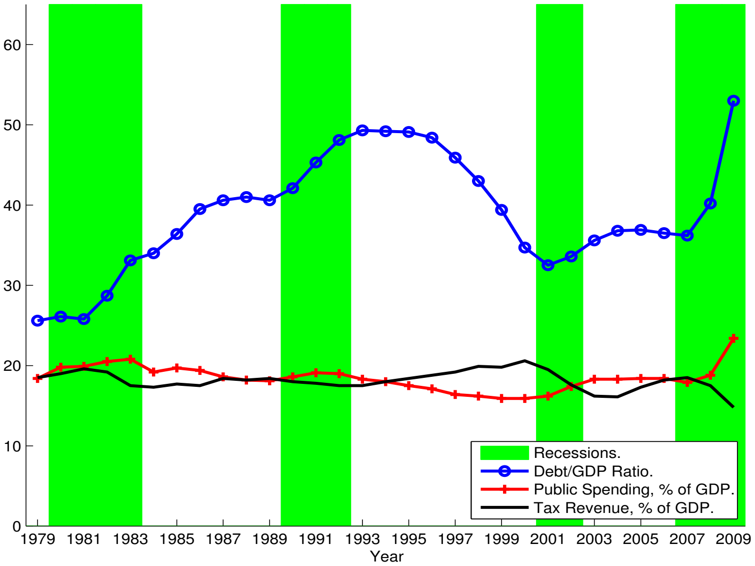

Figure 2: US fiscal variables in booms and recessions, 1979-2009.

with the inequality strict when b is sufficiently low.

Why when the inequality in equation (23) is strict does the mwc not find it optimal to raise taxes and reduce debt in order to equalize the current MCPF with the expected future MCPF? The answer is that if next period's mwc is providing pork, the correspondent increase in revenues will simply be diverted toward pork. This creates a wedge between the current MCPF and the expected future MCPF. The generality of this intuition suggests that a similar result would be true in any dynamic political economy model of debt.

6 Quantitative Analysis

This section provides a quantitative assessment of the theoretical model. The model is calibrated to the U.S. economy and its predictions compared to the data. We study the performance of the model in two main areas. First, explaining the distribution of debt. Second, explaining the cyclical behavior of debt, taxes, and public spending, which includes their volatility, autocorrelation, and correlation with output.

6.1 Empirical facts

We study the U.S. economy over the period 1979 to 2009. Earlier data is not considered for two reasons. First, we would like to abstract from the consequences of World War II and the Korean War, which had significant fiscal impacts not related to business cycle fluctuations. Second, we would like to avoid the high inflation episodes during the 1970s. Since most of government debt is nominal, inference about the behavior of debt is complicated during the periods of high inflation and of high uncertainty about inflation.

Figure 2 and Table 1 present an overview of the behavior of debt, taxes, and public spending during the relevant time period.[30] Two features clearly emerge. First, the debt/GDP ratio is persistently higher than 25% and, on average, is equal to 38.9%. Second, there is high volatility and strong countercyclically of debt. As reported in Table 1, the correlation between debt and GDP is negative and statistically significant. Public spending is 3.2 times less volatile than debt. The tax rate is also not very volatile. The correlations of these two variables with GDP are positive, but statistically insignificant. The autocorrelations of all fiscal variables are very high; the autocorrelation of debt and public spending exceeds that of output, which is equal to 0.87. In sum, the evidence is consonant with the hypothesis that public debt is aggressively used to smooth the impact of business cycle fluctuations on taxation and public good provision. However, public spending and taxes do not exhibit strong cyclical behavior, which is consistent with the findings of the empirical literature discussed in Section 2.

Std			Correlation with output			Autocorrelation		
Debt	Spending	$\frac{\text{Revenue}}{\text{GDP}}$	Debt	Spending	$\frac{\text{Revenue}}{\text{GDP}}$	Debt	Spending	$\frac{\text{Revenue}}{\text{GDP}}$
5.53^{+}	1.74^{+}	1.15	-0.81***	0.10	0.26	0.92***	0.92***	0.73***

Table 1. Empirical second moments of fiscal variables.[31]

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[30] We measure debt as total outstanding federal debt not held by government accounts, taxes as the total federal revenue/GDP ratio, and public spending as total federal expenditures less net interest on debt. Our data sources and detailed definitions of fiscal variables are provided in the Appendix. The Appendix also reports alternative measures of the behavior of the fiscal variables over the cycle.

[31] All data are linearly detrended. Variables marked by $^{+}$ are measured in percent of (average) GDP. The superscript *** is used to denote correlations that are statistically significant at the 1% level.

6.2 Model parameterization

We set the discount factor ρ to 0.95, which is a common choice in the RBC literature (Cooley and Prescott (1995)). This implies that the annual interest rate on bonds ρ is 5.26%. The elasticity of labor supply, ε, is set to 2, which is in a mid-range of parameters used in the literature (see, for example, Greenwood, Hercowitz, and Huffman (1988)). This is also the value used in Aiyagari et al. (2002). The parameters w_L, p, and n are scale parameters that do not directly affect any of our results. We set the first two to 1, and the third to 100. The parameter q is set to 51, implying that the legislature operates by simple majority rule.[32]

	std(GDP)	$\frac{\text{Debt}}{\text{GDP}}$	$\frac{\text{Spending}}{\text{GDP}}$	Avg. length of a recession	% time in a recession
Data	3.35	38.9	18.4	3	1/3
Model	3.29	40.1	18.3	3	1/3

Table 2. Calibration: matching moments.

We calibrate five parameters that are specific to the model: the persistence of a boom, α_{HH}, and a recession, α_{LL}; the productivity of the economy in a boom w_H, and the two parameters governing the relative value and elasticity of the public good, A and σ. The persistence parameters are chosen to match the average frequency and length of recessions in the data. To this end, the probability of transiting from a boom to a recession and from a recession to a boom are respectively chosen to be 16.7% and 33.3%. The remaining parameters w_H, A, and σ are jointly chosen to minimize the distance between the model generated and empirical values of three variables: the standard deviation of (linearly detrended) output, the average debt/GDP ratio, and the average public spending/GDP ratio. Our search, performed over a fine grid, yields the following parameter values: $w_H = 1.0225$, $A = .684$, and $\sigma = 1.52$. As Table 2 reports, the model comes close in matching the targeted moments. In addition, the average tax rate in the model is 20.3% which is close to the 18.1% average in the data.

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[32] Our results are similar for choices of q in the range 51 to 60.

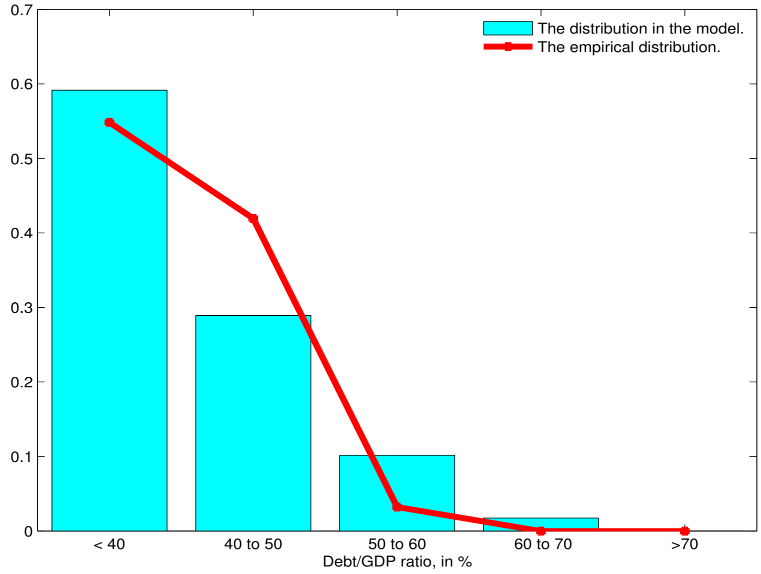

Figure 3: The distribution of the debt/GDP ratio.

6.3 Results

We begin by discussing the distribution of debt. Figure 3 compares the long run distribution of the debt/GDP ratio as well as its empirical counterpart. The average debt/GDP ratio, explicitly targeted in the calibration, is close to that in the data. Though not explicitly targeted, the standard deviation of this variable is also close to its empirical counterpart: 7.4 in the model versus 6.8. Finally, the lowest level of debt in the model is 32% of GDP. In the data, the debt/GDP ratio never falls below 25.8% (its value in 1981) and has been above 32% since 1983. Thus, the model matches well the empirical distribution of debt.

	Std			Correlation with output			Autocorrelation		
	Debt	Spending	Tax Rate	Debt	Spending	Tax Rate	Debt	Spending	Tax Rate
Model	7.71^{+}	0.26^{+}	0.33	-0.42	0.32	-0.32	0.997	0.996	0.996
Data	5.53^{+}	1.74^{+}	1.15^{++}	-0.81	0.10	0.26	0.921	0.920	0.734

Table 3. Results: second moments.[33]

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[33] Variables marked by $^{+}$ are measured in percent of (average) GDP. Variables marked by $^{++}$ are measured by

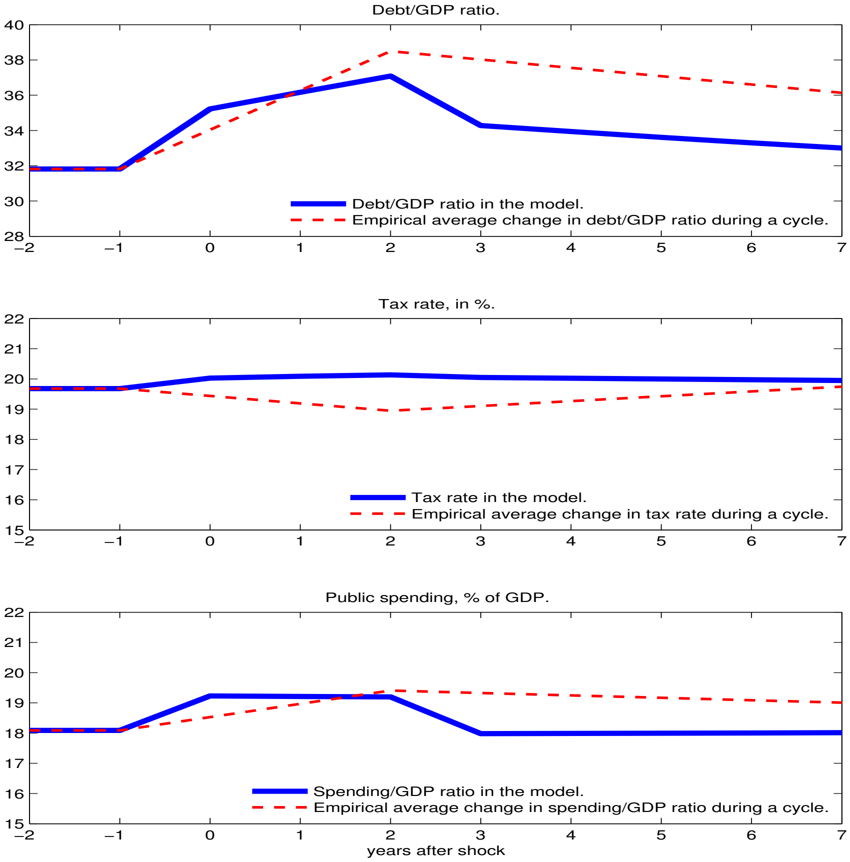
We next turn to the cyclical behavior of fiscal policy variables. Table 3 compares the empirical and model generated second moments of fiscal variables. In both the model and the data, debt is much more volatile than public spending and the tax rate. The volatility of public spending and the tax rate are smaller than their empirical counterparts. This is probably not surprising, since in the model these variables fluctuate only because of technology shocks and policy makers smooth both spending and taxation. In both the model and the data, debt is strongly counter-cyclical. The strength of correlation is higher in the data, with the model picking up about a half of the empirical correlation.[34] Public spending is positively correlated with GDP in both the model and the data, but the correlation is much weaker in the data. Thus, the model's prediction of pro-cyclical public spending is not supported. The tax rate is negatively correlated with GDP in the model, but positively correlated in the data, so the model's prediction of a counter-cyclical tax rate is also not supported. That said, it should be remembered that the positive correlations of taxes and GDP, and spending and GDP in the data are not statistically significant.

Turning to autocorrelation, Table 3 shows that all three fiscal variables exhibit strong persistence in the model. The autocorrelations of debt and public spending are marginally higher than their respective empirical counterparts, while that for the tax rate is about 30% higher than in the data. These autocorrelations are not simply an artifact of a very persistent output process. In fact, because of our two state technology process, the autocorrelation of output is about 0.60, less than it is in the data. Rather the high persistence of the fiscal variables has to do with the smoothing of taxes and public good spending.

As a final exercise to illustrate the predictions of the model, we study how the economy reacts to a recession. In our policy experiment, we assume that the economy has been in a boom long enough to converge to the endogenous lower bound of debt, x_H^*. Then, at date zero a recession occurs, which lasts three years. After that, the economy returns to the high productivity state. Absent a new recession, debt will converge back to its lower bound. Figure 4 presents the evolution of the debt/GDP ratio, the tax rate, and the spending/GDP ratio in this hypothetical scenario.

$\frac{\text{Revenue}}{\text{GDP}}$.

[34] Since, in our theory, the movements in debt are driven by productivity shocks, the reader may ask why we do not match more closely the correlation between debt and GDP. We suspect this is primarily driven by our assumption about the two-state nature of the technology shock: because of tax smoothing the model has essentially two levels of output: high and low, while debt is continuously increasing in recessions, and continuously decreasing in booms until it reaches its lower bound. These non-linear patterns may mechanically result in a lower correlation between debt and output than would arise in a model with a continuous state space and less often binding lower bound on debt.


Debt/GDP ratio.

Tax rate, in %.

Public spending, % of GDP.

Figure 4: A recession followed by a boom.

The dotted lines highlight the corresponding average changes of the policies in the data in a recession and after a recession.

Over the course of the recession, the debt/GDP ratio increases by roughly 5 percentage points, which is about two-thirds of the empirical average increase in the debt/GDP ratio during recessions. While not shown on the Figure, the debt level increases by 8% of its pre-recession value, which is also roughly two-thirds of its empirical counterpart. The responses of the tax rate and public spending are muted. In the model, both the tax rate and the spending/GDP ratio slowly increase during the recession. In the data, the tax rate slowly decreases in recessions, while the spending/GDP ratio increases. After the recession ends, the fiscal variables slowly return to their pre-recession levels.

7 Conclusion

This paper has extended the political economy theory of fiscal policy proposed in Battaglini and Coate (2008) to shed light on the cyclical behavior of fiscal policy. This has required replacing public spending shocks with productivity shocks and making these shocks persistent as opposed to independent and identically distributed. While persistent shocks complicate the characterization of equilibrium, the model remains tractable. In particular, equilibrium policy choices continue to solve a "constrained" planning problem. The difference is that persistence makes the lower bound constraint on debt state-contingent and this is key to understanding the cyclical behavior of fiscal policy.

The theory yields three central predictions. First, in the long run, debt displays a counter-cyclical pattern, increasing in recessions and decreasing in booms. A pro-cyclical debt expansion can only arise the first time the economy moves from recession to boom. This is because any increase in debt has permanent effects on public finances. Second, public spending displays a pro-cyclical pattern, with spending increasing in booms and decreasing in recessions, while tax rates display a counter-cyclical pattern decreasing in booms and increasing in recessions. The equilibrium changes in public spending and taxes therefore serve to amplify the business cycle. Third, equilibrium fiscal policies are such that the marginal cost of public funds obeys a submartingale.

The paper has assessed the quantitative implications of the theory by calibrating the model to the U.S. economy. Despite its parsimonious structure, the model matches well the empirical distribution of debt and also its high volatility, strong persistence, and negative correlation with output. We are not aware of any other theoretical model that is able to quantitatively explain the business cycle properties of debt. The success of the model in explaining the cyclical behavior of public spending and the tax rate is more modest. Consistent with the data, the model implies that these fiscal variables are persistent and not very volatile. However, the predictions of pro-cyclical spending and counter-cyclical taxation do not find empirical support.

There are many ways that the analysis in this paper might be developed in future work. The counter-factual predictions concerning taxation may be resolved in a model with concave utility over consumption. In such a model, policy makers would be concerned also with consumption smoothing, which might suggest cutting taxes in recessions to spur output, rather than increasing them to generate revenue. It would also be useful to relax the assumption that all variation in the economy is caused by a single shock. This assumption required our quantitative exercise to rely on unconditional moments, as in the classical real business cycle literature. Enriching the model to allow for additional shocks could allow us to study more nuanced questions and to better assess the role of political economy distortions in shaping the cyclical behavior of fiscal policy.

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8 Appendix

8.1 Definition of Equilibrium

As background for the analysis in this Apendix, it will be useful to have a more precise definition of political equilibrium. An equilibrium is described by a collection of proposal functions $\{\tau_\theta^r(b),$ $g_\theta^r(b),$ $x_\theta^r(b),$ $s_\theta^r(b)\}_{r=1}^T$ which specify the proposal made by the proposer in round r of the meeting in a period in which the state is (b, θ). Here $\tau_\theta^r(b)$ is the proposed tax rate, $g_\theta^r(b)$ is the public good level, $x_\theta^r(b)$ is the new level of public debt, and $s_\theta^r(b)$ is a transfer offered to the districts of $q-1$ randomly selected representatives. The proposer's district receives the surplus revenues $B_\theta(\tau_\theta^r(b), g_\theta^r(b), x_\theta^r(b); b) - (q-1)s_\theta^r(b)$. Associated with any equilibrium are a collection of value functions $\{v_\theta^r(b)\}_{r=1}^{T+1}$ which specify the expected future payoff of a legislator at the beginning of proposal round r in a period in which the state is (b, θ). In what follows, we will drop the superscript and refer to the round 1 value function as $v_\theta(b)$ and the round 1 policy proposal as $\{\tau_\theta(b),$ $g_\theta(b),$ $x_\theta^r(b),$ $s_\theta(b)\}$.

In equilibrium, there is a reciprocal feedback between the policy proposals $\{\tau_\theta^r(b),$ $g_\theta^r(b),$ $x_\theta^r(b),$ $s_\theta^r(b)\}_{r=1}^T$ and the associated value functions $\{v_\theta^r(b)\}_{r=1}^{T+1}$. On the one hand, given that future payoffs are described by the value functions, the prescribed policy proposals must maximize the proposer's payoff subject to the incentive constraint of getting the required number of affirmative votes and the appropriate feasibility constraints. Formally, given $\{v_\theta^r(b)\}_{r=1}^{T+1}$, for each proposal round r and state (b, θ), the proposal $\{\tau_\theta^r(b),$ $g_\theta^r(b),$ $x_\theta^r(b),$ $s_\theta^r(b)\}$ must solve the problem:

$$\max_{(\tau,g,x,s)} \left\{ \begin{array}{c} u_\theta(\tau, g) + B_\theta(\tau, g, x; b) - (q-1)\, s + \beta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \\ \\ s.t. \quad u_\theta(\tau, g) + s + \beta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \geq v_\theta^{r+1}(b), \\ \\ B_\theta(\tau, g, x; b) \geq (q-1)s, \quad s \geq 0 \;\&\; x \in [\underline{x}, \overline{x}]. \end{array} \right\} \tag{24}$$

The first constraint is the incentive constraint and the remainder are feasibility constraints. The formulation reflects the assumption that on the equilibrium path, the proposal made in round 1 is accepted.

On the other hand, the value functions $\{v_\theta^r(b)\}_{r=1}^{T+1}$ are themselves determined by the equilibrium policy proposals. For proposal rounds $r = 1, ..., T$, the legislators' round r value functions $v_L^r(b)$ and $v_H^r(b)$ are determined recursively using $\{\tau_\theta^r(b),$ $g_\theta^r(b),$ $x(b),$ $s_\theta^r(b)\}_{r=1}^T$ by:

$$v_\theta^r(b) = u_\theta(\tau_\theta^r(b), g_\theta^r(b)) + \frac{B_\theta(\tau_\theta^r(b), g_\theta^r(b), x_\theta^r(b); b)}{n} + \delta[\alpha_{\theta H} v_H(x_\theta^r(b)) + \alpha_{\theta L} v_L(x_\theta^r(b))] \tag{25}$$

for $\theta \in \{L, H\}$. To understand this condition, note that the second term in (25) is the expected benefit of pork transfers. Pork transfers clearly depend on who the proposer will be and on his choice of coalition: in a symmetric equilibrium, however, legislators receive the same expected amount, $\frac{1}{n}B(\tau^r_\theta(b), g^r_\theta(b), x^r_\theta(b); b)$. Recall that if the round T proposal is rejected, the assumption is that a legislator is appointed to choose a default tax rate, public good level, level of debt and a uniform transfer. Thus,

$$v^{T+1}_\theta(b) = \max_{(\tau,g,x)} \left\{ u_\theta(\tau, g) + \frac{B_\theta(\tau, g, x; b)}{n} + \delta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] : B_\theta(\tau, g, x; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \right\}. \tag{26}$$

Definition. *A political equilibrium consists of a collection of policy functions* $\{\tau^r_\theta(b),\ g^r_\theta(b),\ x^r_\theta(b),\ s^r_\theta(b)\}^T_{r=1}$ *and value functions* $\{v^r_\theta(b)\}^{T+1}_{r=1}$ *such that* $\{\tau^r_\theta(b),\ g^r_\theta(b),\ x^r_\theta(b),\ s^r_\theta(b)\}$ *solves (24) given* $v^{r+1}_\theta(b)$ *for all r;* $v^r_\theta(b)$ *satisfies (25) given* $\{\tau^r_\theta(b),\ g^r_\theta(b),\ x^r_\theta(b),\ s^r_\theta(b)\}$*; and* $v^{T+1}_\theta(b)$ *satisfies (26).*

8.2 Proof of Proposition 1

To prove Proposition 1 we start with an auxiliary result.

Lemma A.1. *Suppose that the value functions* $v_H(b)$ *and* $v_L(b)$ *solve the system of functional equations (19) where* x^*_L *and* x^*_H *satisfy (18). Then, there exists an equilibrium in which the round 1 value functions are* $v_H(b)$ *and* $v_L(b)$ *and the round 1 policy choices* $\{\tau_\theta(b),\ g_\theta(b),\ x^r_\theta(b)\}$ *are the optimal policy functions for program (19).*

Proof. Let $\widetilde{v}_H$ and $\widetilde{v}_L$ be a pair of value functions and $\widetilde{x}_H$ and $\widetilde{x}_L$ a pair of debt levels such that (i) $\widetilde{v}_H$ and $\widetilde{v}_L$ solve (19) given $\widetilde{x}_H$ and $\widetilde{x}_L$, and, (ii) $\widetilde{x}_H$ and $\widetilde{x}_L$ solve (18) given $\widetilde{v}_H$ and $\widetilde{v}_L$. Let $(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b))$ be the corresponding optimal policies that solve the program in (19). For each proposal round r and state (b, θ) define the following strategies:

$$(\tau^r_\theta(b), g^r_\theta(b), x^r_\theta(b)) = (\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b));$$

and for proposal rounds $r = 1, ..., T-1$

$$s^r_\theta(b) = \left\{ \begin{array}{c} B_\theta(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b); b)/n \quad \text{if } r = 1, ..., T-1 \\ \\ v^{T+1}_\theta(b) - u_\theta(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b)) - \beta\left[\alpha_{\theta H}\widetilde{v}_H(\widetilde{x}_\theta(b)) + \alpha_{\theta L}\widetilde{v}_L(\widetilde{x}_\theta(b))\right] \quad \text{if } r = T \end{array} \right. ;$$

where

$$v^{T+1}_\theta(b) = \max_{(\tau,g,x)} \left\{ \begin{array}{c} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta\left[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)\right] \\ \\ \text{s.t. } B_\theta(\tau, g, x; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\}.$$

Given these proposals, the legislators' round one value functions are given by $\widetilde{v}_H$ and $\widetilde{v}_L$. This follows from the fact that

$$v_\theta^1(b) = u_\theta(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b)) + \frac{B_\theta(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b), \widetilde{b}'_\theta(b); b)}{n} + \beta[\alpha_{\theta H}\widetilde{v}_H(\widetilde{x}_\theta(b)) + \alpha_{\theta L}\widetilde{v}_L(\widetilde{x}_\theta(b))] = \widetilde{v}_\theta(b).$$

Similarly, the round $r = 2, ..., T$ legislators' value functions are given by $\widetilde{v}_H$ and $\widetilde{v}_L$.

To show that $\{\tau_\theta^r(b), g_\theta^r(b), x_\theta^r(b), s_\theta^r(b)\}_{r=1}^T$ together with the associated value functions $\{v_\theta^r(b)\}_{r=1}^{T+1}$ describe an equilibrium, we need only show that for proposal rounds $r = 1, .., T$ the proposal $\{\tau_\theta^r(b), g_\theta^r(b), x_\theta^r(b), s_\theta^r(b)\}$ solves the problem

$$\begin{aligned}
&\max_{(\tau,g,x,s)} u_\theta(\tau, g) + B_\theta(\tau, g, x; b) - (q-1)\, s + \beta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \\
&\quad s.t. \quad u_\theta(\tau, g) + s + \beta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \geq \Upsilon_\theta^{r+1}(b) \\
&\qquad\qquad B_\theta(\tau, g, x; b) \geq (q-1)s, \quad s \geq 0 \ \&\ x \in [\underline{x}, \overline{x}],
\end{aligned}$$

where $\Upsilon_\theta^{r+1}(b) = \widetilde{v}_\theta(b)$ for $r = 1..., T-1$, and $\Upsilon_\theta^{T+1}(b) = v_\theta^{T+1}(b)$. We show this result only for $r = 1, ..., T-1$ – the argument for $r = T$ being analogous.

Consider some proposal round $r = 1, ..., T-1$. Let (b, θ) be given. To simplify notation, let

$$(\widehat{\tau}, \widehat{g}, \widehat{x}, \widehat{s}) = (\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b), \frac{B_\theta(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b), \widetilde{x}_\theta(b); b)}{n}).$$

Since $\widetilde{x}_\theta$ solves (18), it follows from the discussion in Section 5.1 (and it can easily be formally verified) that $(\widehat{\tau}, \widehat{g}, \widehat{x})$ solves the problem:

$$\begin{aligned}
&\max_{(\tau,g,x)} u_\theta(\tau, g) + \tfrac{B_\theta(\tau,g,x;b)}{q} + \beta[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)] \\
&\quad s.t. \quad B_\theta(\tau, g, x; b) \geq 0 \ \&\ x \in [\underline{x}, \overline{x}],
\end{aligned}$$

and that

$$\widehat{s} = \widetilde{v}_\theta(b) - u_\theta(\widehat{\tau}, \widehat{g}) - \beta[\alpha_{\theta H}\widetilde{v}_H(\widehat{x}) + \alpha_{\theta L}\widetilde{v}_L(\widehat{x})].$$

Suppose that $(\widehat{\tau}, \widehat{g}, \widehat{x}, \widehat{s})$ does not solve the round r proposer's problem. Then there exist some (τ', g', x', s') which achieves a higher value of the proposer's objective function. We know that $s' \geq \widetilde{v}_\theta(b) - u_\theta(\tau', g') - \beta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')]$. Thus, we have that the value of the proposer's objective function satisfies

$$\begin{aligned}
& u_\theta(\tau', g') + B_\theta(\tau', g', x'; b) - (q-1)\, s' + \beta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')] \\
\leq\ & q\{u_\theta(\tau', g') + \beta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')]\} + B_\theta(\tau', g', x'; b).
\end{aligned}$$

But since $B_\theta(\tau', g', x'; b) \geq 0$, we know that

$$\begin{aligned} & q\{u_\theta(\tau', g') + \beta[\alpha_{\theta H}\widetilde{v}_H(x') + \alpha_{\theta L}\widetilde{v}_L(x')]\} + B_\theta(\tau', g', x'; b) \\ \leq \quad & q\{u_\theta(\widehat{\tau}, \widehat{g}) + \beta[\alpha_{\theta H}\widetilde{v}_H(\widehat{x}) + \alpha_{\theta L}\widetilde{v}_L(\widehat{x})]\} + B_\theta(\widehat{\tau}, \widehat{g}, \widehat{x}; b). \end{aligned}$$

But the right hand side of the inequality is the value of the proposer's objective function under the proposal $(\widehat{\tau}, \widehat{g}, \widehat{x}, \widehat{s})$. This therefore contradicts the assumption that (τ', g', x', s') achieves a higher value for the proposer's problem. ■

By Lemma A.1, we can establish the existence of an equilibrium by showing that we can find a pair of value functions $v_H(b)$ and $v_L(b)$ and a pair of debt thresholds b_L^* and b_H^* such that (i) $v_H(b)$ and $v_L(b)$ solve (19) given x_L^* and x_H^*, and, (ii) x_L^* and x_H^* solve (18) given $v_H(b)$ and $v_L(b)$. We simply sketch how to do this here, the details are available on request.

Let F denote the set of all real valued functions $v(\cdot)$ defined over the set $[\underline{x}, \overline{x}]$ that are continuous and concave. For each $\theta \in \{H, L\}$ and any $z_\theta \in [(R_L(\tau^*) - pg^*)/\rho, \overline{x}]$, define the operator $T_{z_\theta}^\theta : F \times F \rightarrow F$ as follows:

$$T_{z_\theta}^\theta(v_H, v_L)(b) = \left\{ \begin{array}{c} \max_{(\tau,g,x)} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta\left[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)\right] \\ \\ s.t. \quad B_\theta(\tau, g, x; b) \geq 0,\ \tau \geq \tau^*,\ g \leq g^* \ \&\ x \in [z_\theta, \overline{x}] \end{array} \right\}.$$

Let $\mathbf{z} = (z_H, z_L)$ and let $T_\mathbf{z}(v_H, v_L)(b) = (T_{z_H}^H(v_H, v_L)(b), T_{z_L}^L(v_H, v_L)(b))$ be the corresponding function from $F \times F$ to itself. For any $\mathbf{z} \in [(R_L(\tau^*) - pg^*)/\rho, \overline{x}]^2$, it can be verified that $T_\mathbf{z}$ is a contraction and admits a unique fixed point $v_\mathbf{z}$ (where we use the subscript $\mathbf{z}$ to indicate that this fixed point depends on $\mathbf{z}$). Given $v_\mathbf{z}$, let

$$M_\theta(\mathbf{z}) = \arg\max\{\frac{x}{q} + \beta\left[\alpha_{\theta H} v_{H\mathbf{z}}(x) + \alpha_{\theta L} v_{L\mathbf{z}}(x)\right] : x \in [z_\theta, \overline{x}]\}$$

and let $M(\mathbf{z}) = M_H(\mathbf{z}) \times M_L(\mathbf{z})$. Then, we have an equilibrium if we can find a fixed point of this correspondence, $\mathbf{z} \in M(\mathbf{z})$. This can be proven by showing that M satisfies the conditions of *Kakutani's Fixed Point Theorem.* ■

8.3 Proof of Proposition 2

Let $\{\tau_\theta^r(b),\, g_\theta^r(b),\, x_\theta^r(b),\, s_\theta^r(b)\}_{r=1}^T$ be an equilibrium with associated value functions $\{v_\theta^r(b)\}_{r=1}^{T+1}$. It is enough to show that for $\theta \in \{L, H\}$, $\{\tau_\theta(b),\, g_\theta(b),\, x_\theta(b)\}$ solves the problem

$$\begin{array}{c} \max_{(\tau,g,x)} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t.\;\; B_\theta(\tau, g, x; b) \geq 0,\, \tau \geq \tau^*,\, g \leq g^* \;\&\; x \in [x_\theta^*, \overline{x}], \end{array} \tag{27}$$

where x_θ^* satisfies (18). For then it would follow immediately from (25) that the value functions $v_L(b)$ and $v_H(b)$ have the required properties.

We begin by making precise the claim made at the beginning of Section 5.1 that, given transferable utility, the proposer is effectively making decisions to maximize the collective utility of q legislators under the assumption that they get to divide any surplus revenues among their districts.

Lemma A.2. *Let $\{\tau_\theta^r(b),\, g_\theta^r(b),\, x_\theta^r(b),\, s_\theta^r(b)\}_{r=1}^T$ be an equilibrium with associated value functions $\{v_\theta^r(b)\}_{r=1}^{T+1}$. Then, for all states (b, θ), the tax rate-public good-public debt triple $(\tau_\theta^r(b), g_\theta^r(b), x_\theta^r(b))$ proposed in any round r solves the problem*

$$\begin{array}{c} \max_{(\tau,g,x)} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{q} + \beta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t. \quad B_\theta(\tau, g, x; b) \geq 0 \;\&\; x \in [\underline{x}, \overline{x}]\,. \end{array} \tag{28}$$

Moreover, the transfer to coalition members is given by

$$s_\theta^r(b) = v_\theta^{r+1}(b) - u_\theta(\tau_\theta^r(b), g_\theta^r(b)) - \beta\left[\alpha_{\theta L} v_L(x_\theta^r(b)) + \alpha_{\theta H} v_H(x_\theta^r(b))\right].$$

Proof: The proof of this result is similar to the proof of an analogous result in Battaglini and Coate (2008) and thus is omitted. A proof is available from the authors upon request.

As we argued in the text, if the constraint $B_\theta(\tau, g, x; b) \geq 0$, is not binding, then the solution to problem (28) is $(\tau^*, g^*, x_\theta^*)$. On the other hand, if the constraint is binding, then the solution to this problem solves the problem

$$\begin{array}{c} \max_{(\tau,g,x)} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t. \quad B_\theta(\tau, g, x; b) \geq 0 \;\&\; x \in [\underline{x}, \overline{x}]\,. \end{array} \tag{29}$$

Letting b_θ^* be as defined in (20), we conclude that $\{\tau_\theta(b),\, g_\theta(b),\, x_\theta(b)\} = (\tau^*, g^*, x_\theta^*)$ when $b \leq b_\theta^*$ and solves (29) when $b > b_\theta^*$. Thus, we need to show (i) that when $b \leq b_\theta^*$ the solution to problem

(27) is $(\tau^*, g^*, x^*_\theta)$, and, (ii) that when $b > b^*_\theta$ the constraints $\tau \geq \tau^*$, $g \leq g^*$ and $x \geq x^*_\theta$ will not be binding in problem (27). For (ii), note first that the solution to (29) when $b = b^*_\theta$ is $(\tau^*, g^*, x^*_\theta)$ and second that the optimal tax rate and debt level for problem (29) are non decreasing in b and the public good is non increasing in b. For (i), note that when $b \leq b^*_\theta$ the budget constraint cannot be binding in problem (27) and, if the budget constraint is not binding, the individual constraints $\tau \geq \tau^*$, $g \leq g^*$ and $x \geq x^*_\theta$ must all bind. ■

8.4 Proof of Lemma 1

From Proposition 2, we know that

$$v_\theta(b) = \max_{(\tau,g,x)} \left\{ \begin{array}{c} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta L} v_L(x) + \alpha_{\theta H} v_H(x)] \\ s.t.\;\; B_\theta(\tau, g, x; b) \geq 0,\, \tau \geq \tau^*,\, g \leq g^* \;\&\; x \in [x^*_\theta, \overline{x}] \end{array} \right\}.$$

Moreover, from the discussion in the text, we know that if $b \leq b^*_\theta$ the optimal policies are $(\tau^*, g^*, x^*_\theta)$, and, if $b > b^*_\theta$, the constraints $\tau \geq \tau^*$, $g \leq g^*$ and $x \geq x^*_\theta$ in the maximization problem will not be binding, but the budget constraint will be binding.

Suppose first that $b_o < b^*_\theta$. Then, we know that in a neighborhood of b_o it must be the case that

$$v_\theta(b) = u_\theta(\tau^*, g^*) + \frac{B_\theta(\tau^*, g^*, x^*_\theta; b)}{n} + \beta[\alpha_{\theta H} v_H(x^*_\theta) + \alpha_{\theta L} v_L(x^*_\theta)].$$

Thus, it is immediate that the value function $v_\theta(b)$ is differentiable at b_o and that

$$v'_\theta(b_o) = -(\frac{1+\rho}{n}).$$

Now suppose that $b_o > b^*_\theta$. Then, we know that in a neighborhood of b_o it must be the case that

$$v_\theta(b) = \max_{(\tau,g,x)} \left\{ \begin{array}{c} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta\left[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)\right] \\ B_\theta(\tau, g, x; b) \geq 0 \;\&\; x \in [x^*_\theta, \overline{x}] \end{array} \right\}.$$

Define the function

$$g(b) = \frac{R_\theta(\tau_\theta(b_o)) + x_\theta(b_o) - (1+\rho)b}{p}$$

and let

$$\varphi(b) = u_\theta(\tau_\theta(b_o), g(b)) + \frac{B_\theta(\tau_\theta(b_o), g(b), x(b_o); b)}{n} + \beta\left[\alpha_{\theta H} v_H(x(b_o)) + \alpha_{\theta L} v_L(x(b_o))\right].$$

Notice that $(\tau_\theta(b_o), g(b), x_\theta(b_o))$ is a feasible policy when the initial debt level is b so that in a neighborhood of b_o we must have that $v_\theta(b) \geq \varphi(b)$. Moreover, $\varphi(b)$ is twice continuously differentiable with derivatives

$$\varphi'(b) = -Ag(b)^{-\sigma}(\frac{1+\rho}{p}),$$

and

$$\varphi''(b) = -\sigma Ag(b)^{-\sigma-1}(\frac{1+\rho}{p})^2 < 0.$$

The second derivative property implies that $\varphi(b)$ is strictly concave. It follows from Theorem 4.10 of Stokey, Lucas and Prescott (1989) that $v_\theta(b)$ is differentiable at b_o with derivative $v_\theta'(b) = \varphi'(b_o) = -Ag_\theta(b_o)^{-\sigma}(\frac{1+\rho}{p})$. To complete the proof note that $(\tau_\theta(b_o), g_\theta(b_o))$ must solve the problem:

$$\max_{(\tau,g)} \left\{ \begin{array}{c} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x_\theta(b_o);b_o)}{n} \\ \\ B_\theta(\tau, g, x_\theta(b_o); b_o) \geq 0 \end{array} \right\},$$

which implies that $nAg_\theta(b_o)^{-\sigma} = p[\frac{1-\tau_\theta(b_o)}{1-\tau_\theta(b_o)(1+\varepsilon)}]$. Thus, we have that

$$v_\theta'(b) = -[\frac{1-\tau_\theta(b_o)}{1-\tau_\theta(b_o)(1+\varepsilon)}](\frac{1+\rho}{n}).$$

■

8.5 Proof of Lemma 2

We begin by showing that $x_H^* \geq x_L^*$. Suppose that, to the contrary, that $x_H^* < x_L^*$. There are two possibilities. The first is that $b_L^* < b_H^*$. In this case, it follows from the first order condition

$$\frac{1}{q} = -\beta[\alpha_{\theta H} v_H'(x_\theta^*) + \alpha_{\theta L} v_L'(x_\theta^*)], \tag{30}$$

and Lemma 1 that $b_L^* < x_H^* < x_L^* \leq b_H^*$ and that x_H^* and x_L^* satisfy the following two first order conditions:

$$\alpha_{HL}(\frac{1-\tau_L(x_H^*)}{1-\tau_L(x_H^*)(1+\varepsilon)}) + \alpha_{HH} = \frac{n}{q},$$

and

$$\alpha_{LL}(\frac{1-\tau_L(x_L^*)}{1-\tau_L(x_L^*)(1+\varepsilon)}) + \alpha_{LH} \leq \frac{n}{q} \ (= \text{if } x_L^* < b_H^*).$$

But since $x_H^* < x_L^*$ we know that

$$\frac{1-\tau_L(x_H^*)}{1-\tau_L(x_H^*)(1+\varepsilon)} < \frac{1-\tau_L(x_L^*)}{1-\tau_L(x_L^*)(1+\varepsilon)}.$$

In addition, $\alpha_{HL} \leq \alpha_{LL}$ and hence the above two first order conditions are clearly inconsistent.

The second possibility is that $b_L^* > b_H^*$. In this case, it follows from (30) and Lemma 1 that $b_H^* < x_H^* < x_L^* \leq b_L^*$. Since $x_H^* > b_H^*$, it must be that in a boom with debt level $b = x_H^*$ the policy is such that $\tau_H(x_H^*) > \tau^*$, $g_H(x_H^*) < g^*$, and $x_H(x_H^*) > x_H^*$. This implies that

$$\begin{aligned} 0 &= B_H(\tau_H(x_H^*), g_H(x_H^*), x_H(x_H^*); x_H^*) \\ &> B_H(\tau^*, g^*, x_H^*; x_H^*) = R_H(\tau^*) - pg^* - \rho x_H^* > R_H(\tau^*) - pg^* - \rho x_L^*. \end{aligned} \tag{31}$$

Since $x_L^* \leq b_L^*$, it must be that in a recession with debt level $b = x_L^*$, the policy is such that $\tau_L(x_L^*) = \tau^*$, $g_L(x_L^*) = g^*$, and $x_L(x_L^*) = x_L^*$. This implies:

$$0 \leq B_L(\tau^*, g^*, x_L^*; x_L^*) = R_L(\tau^*) - pg^* - \rho x_L^* < R_H(\tau^*) - pg^* - \rho x_H^*,$$

which is in contradiction with (31).

Given that $x_H^* \geq x_L^*$, it follows from (20) that $b_L^* < b_H^*$. Lemma 1 and the first order condition (30) imply that $x_H^* \leq b_H^*$ and that $b_L^* < x_L^*$. ■

8.6 Proof of the results of Section 5.1

In Section 5.1 we claim that (i) if $b \geq b_H^*$ then $\tau_L(b) > \tau_H(b)$, $g_L(b) < g_H(b)$ and $x(b) > x(b)$; and (ii) there is a $\widehat{b} \in (b_L^*, b_H^*)$ such that new debt will be higher in a recession than a boom if and only if $b > \widehat{b}$. Assume for now that we know that when $b \geq b_H^*$, $\tau_L(b) > \tau_H(b)$. Then the remaining components of part (i) follow easily. The first order conditions tell us that $\{\tau_\theta(b), g_\theta(b)\}$ must satisfy the following equality:

$$nAg^{-\sigma} = p[\frac{1-\tau}{1-\tau(1+\varepsilon)}],$$

which implies that $g_L(b) < g_H(b)$. In addition, since $x_H^* \leq b_H^* \leq b$, by Lemma 3 below we have that

$$x_H(b) \leq b < x_L(b).$$

Moreover, part (ii) follows from the facts that $x_L(b)$ is increasing in b on the interval $(b_L^*, b_H^*]$, $x_H(b)$ is constant on the interval $(b_L^*, b_H^*]$, $x_H(b_L^*) > x_L(b_L^*)$, and $x_H(b_H^*) < x_L(b_H^*)$ (by part (i)).

It remains therefore to show that when $b \geq b_H^*$, $\tau_L(b) > \tau_H(b)$. When $b \geq b_H^*$, we know from the discussion following Proposition 2 that $\{\tau_\theta(b),\ g_\theta(b),\ x_\theta(b)\}$ satisfies the following three equations:

$$nAg^{-\sigma} = p[\frac{1-\tau}{1-\tau(1+\varepsilon)}],$$

$$[\frac{1-\tau}{1-\tau(1+\varepsilon)}] = -\beta n[\alpha_{\theta H} v_H'(x) + \alpha_{\theta L} v_L'(x)],$$

and

$$B_\theta(\tau, g, x; b) = 0.$$

Suppose, contrary to the claim in the Lemma, that $\tau_L(b) \leq \tau_H(b)$. Then it follows immediately that $g_L(b) \geq g_H(b)$. In addition, we have that

$$-\beta n[\alpha_{HH} v_H'(x_H(b)) + \alpha_{HL} v_L'(x_H(b))] \geq -\beta n[\alpha_{LH} v_H'(x_L(b)) + \alpha_{LL} v_L'(x_L(b))].$$

Suppose that it were the case that $-v_H'(x_H(b)) \leq -v_L'(x_H(b))$. Then, since $\alpha_{HH} > \alpha_{LH}$, we would have that

$$-\beta n[\alpha_{HH} v_H'(x_H(b)) + \alpha_{HL} v_L'(x_H(b))] \leq -\beta n[\alpha_{LH} v_H'(x_H(b)) + \alpha_{LL} v_L'(x_H(b))].$$

Combining these two inequalities we could conclude that $x_H(b) \geq x_L(b)$. But then we would have

$$0 = B_H(\tau_H(b), g_H(b), x_H(b); b) > B_L(\tau_L(b), g_L(b), x_L(b); b) = 0$$

a contradiction. Thus, we would have shown that $\tau_L(b) > \tau_H(b)$.

It follows that we can prove that $\tau_L(b) > \tau_H(b)$ by showing the following result:

Lemma A.3. *If v_H and v_L are differentiable at $b \in [\underline{x}, \overline{x}]$, then*

$$-v_H'(b) \leq -v_L'(b).$$

Proof of Lemma A.3: As in the proof of Proposition 1, let F denote the set of all real valued functions $v(\cdot)$ defined over the set $[\underline{x}, \overline{x}]$ that are continuous and concave. For $\theta \in \{H, L\}$, define the operator $T^\theta : F \times F \to F$ as follows:

$$T^\theta(v_H, v_L)(b) = \left\{ \begin{array}{c} \max_{(\tau,g,x)} u_\theta(\tau, g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta\left[\alpha_{\theta H} v_H(x) + \alpha_{\theta L} v_L(x)\right] \\ s.t. \quad B_\theta(\tau, g, x; b) \geq 0,\ \tau \geq \tau^*,\ g \leq g^* \ \&\ x \in [x_\theta^*, \overline{x}] \end{array} \right\}.$$

Let $T(v_H, v_L)(b) = (T^H(v_H, v_L)(b), T^L(v_H, v_L)(b))$ be the corresponding function from $F \times F$ to itself. From Proposition 2, we know that $(v_H, v_L) = T(v_H, v_L)$. Moreover, T is a contraction.

Now let $\widetilde{v}_H$ and $\widetilde{v}_L$ belong to F and assume that for any b if ξ_L and ξ_H are sub-gradients of $\widetilde{v}_L$ and $\widetilde{v}_H$ at b, then we have that: $-\xi_L \geq -\xi_H$. Define $\mathbf{v}_0 = (\widetilde{v}_H, \widetilde{v}_H)$ and consider the sequence of functions $\langle v_{\theta k}(b) \rangle_{k=1}^{\infty}$ for $\theta = H, L$, defined inductively as follows: $v_{\theta 1} = T^\theta(\mathbf{v}_0)$, and

$v_{\theta k+1} = T^{\theta}(v_{Hk}, v_{Lk})$. Let $\mathbf{v}_k = (v_{Hk}, v_{Lk})$ and note that, since T is a contraction, $\langle \mathbf{v}_k \rangle_{k=1}^{\infty}$ must converge to (v_H, v_L).

Finally, for all $\mu > 0$, let

$$X_{\theta}^{\mu}(\mathbf{v}_k) = \arg\max_{x} \{\frac{x}{\mu} + \beta\, [\alpha_{\theta H} v_{Hk}(x) + \alpha_{\theta L} v_{Lk}(x)] : x \in [x_{\theta}^*, \overline{x}]\}$$

and let $x_{\theta}^{\mu}(\mathbf{v}_k)$ be the largest element of the compact set $X_{\theta}^{\mu}(\mathbf{v}_k)$. Notice that $x_{\theta}^{\mu}(\mathbf{v}_k)$ is non-increasing in μ. Also let

$$b_{\theta}^*(x) = \frac{R_{\theta}(\tau^*) + x - pg^*}{1+\rho}.$$

Then we have:

Claim: *For all k, for any $b \in [\underline{x}, \overline{x}]$ if ξ_L^k and ξ_H^k are sub-gradients of v_{Lk} and v_{Hk} at b then we have that: $-\xi_L^k \geq -\xi_H^k$. In addition, if $b \in (b_H^*(x_H^q(\mathbf{v}_{k-1})), \overline{x}]$, then v_{Hk} and v_{Lk} are differentiable at b and $-v_{Lk}'(b) > -v_{Hk}'(b)$.*

Proof: The proof proceeds via induction. Consider the claim for $k = 1$. In state θ if (τ, g, x) is a solution to the problem

$$\begin{array}{c} \max u_{\theta}(\tau, g) + \frac{B_{\theta}(\tau,g,x;b)}{n} + \beta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \\ B_{\theta}(\tau, g, x; b) \geq 0,\ g \leq g^*,\ \tau \geq \tau^* \ \&\ x \in [x_{\theta}^*, \overline{x}] \end{array},$$

then: (i) if $b \leq b_{\theta}^*(x_{\theta}^n(\mathbf{v}_0))$, $(\tau, g) = (\tau^*, g^*)$ and $x \in X_{\theta}^n(\mathbf{v}_0) \cap \{x : B_{\theta}(\tau^*, g^*, x; b) \geq 0\}$; (ii) if $b \in (b_{\theta}^*(x_{\theta}^n(\mathbf{v}_0)), b_{\theta}^*(x_{\theta}^q(\mathbf{v}_0))]$, $(\tau, g) = (\tau^*, g^*)$ and $B_{\theta}(\tau^*, g^*, x; b) = 0$; and, (iii) if $b > b_{\theta}^*(x_{\theta}^q(\mathbf{v}_0))$, (τ, g, x) is uniquely defined and the budget constraint is binding. Moreover, $\tau > \tau^*$ and $g < g^*$. Denote the solution in case (iii) as $(\tau_{\theta}(b; \mathbf{v}_0), g_{\theta}(b; \mathbf{v}_0), x_{\theta}(b; \mathbf{v}_0))$.

It follows from this that, if $b \leq b_{\theta}^*(x_{\theta}^n(\mathbf{v}_0))$

$$T^{\theta}(\mathbf{v}_0)(b) = u_{\theta}(\tau^*, g^*) + \frac{B_{\theta}(\tau^*, g^*, x_{\theta}^n(\mathbf{v}_0); b)}{n} + \beta[\alpha_{\theta H}\widetilde{v}_H(x_{\theta}^n(\mathbf{v}_0)) + \alpha_{\theta L}\widetilde{v}_L(x_{\theta}^n(\mathbf{v}_0))].$$

Thus, $T^{\theta}(\mathbf{v}_0)$ is differentiable and its derivative is

$$-\frac{dT^{\theta}(\mathbf{v}_0)(b)}{db} = \frac{1+\rho}{n}.$$

If $b \in (b_{\theta}^*(x_{\theta}^n(\mathbf{v}_0)), b_{\theta}^*(x_{\theta}^q(\mathbf{v}_0))]$, then

$$T^{\theta}(\mathbf{v}_0)(b) = u_{\theta}(\tau^*, g^*) + \beta[\alpha_{\theta H}\widetilde{v}_H(pg^* + (1+\rho)b - R_{\theta}(\tau^*)) + \alpha_{\theta L}\widetilde{v}_L(pg^* + (1+\rho)b - R_{\theta}(\tau^*))].$$

It follows that if μ_θ is a sub-gradient of $T^\theta(\mathbf{v}_0)$ at b there exist sub-gradients ξ_H and ξ_L of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_\theta(\tau^*)$ such that $\mu_\theta = \alpha_{\theta H}\xi_H + \alpha_{\theta L}\xi_L$. Notice that in this range, $b \in (b^*_\theta(x^n_\theta(\mathbf{v}_0)), b^*_\theta(x^q_\theta(\mathbf{v}_0))]$ and hence if μ_θ is a sub-gradient of $T^\theta(\mathbf{v}_0)$ at b

$$-\beta\mu_\theta(1+\rho) \in (\frac{1+\rho}{n}, \frac{1+\rho}{q}].$$

If $b > b^*_\theta(x^q_\theta(\mathbf{v}_0))$ then

$$T^\theta(\mathbf{v}_0)(b) = \max_{(\tau,g,x)} \left\{ \begin{array}{c} u_\theta(\tau,g) + \frac{B_\theta(\tau,g,x;b)}{n} + \beta[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)] \\ \\ B_\theta(\tau,g,x;b) \geq 0 \ \& \ x \in [x^*_\theta, \overline{x}] \end{array} \right\}.$$

By the same argument used to prove Lemma 1, $T^\theta(\mathbf{v}_0)$ is differentiable and its derivative is

$$-\frac{dT^\theta(\mathbf{v}_0)(b)}{db} = \frac{1-\tau_\theta(b;\mathbf{v}_0)}{n(1-\tau_\theta(b;\mathbf{v}_0)(1+\varepsilon))}(1+\rho).$$

Since $\tau_\theta(b;\mathbf{v}_0) > \tau^*$, in this range we have that

$$-\frac{dT^\theta(\mathbf{v}_0)(b)}{db} > \frac{(1+\rho)}{q}.$$

Given the expressions for the derivatives and subgradients derived above, the result would follow for $k = 1$ if: (i) $b^*_L(x^n_L(\mathbf{v}_0)) \leq b^*_H(x^n_H(\mathbf{v}_0))$; (ii) $b^*_L(x^q_L(\mathbf{v}_0)) \leq b^*_H(x^q_H(\mathbf{v}_0))$; (iii) for all $b \in (b^*_H(x^n_H(\mathbf{v}_0)), b^*_L(x^q_L(\mathbf{v}_0)))$ if ξ_H and ξ_L are subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^*+(1+\rho)b-R_H(\tau^*)$ and ξ'_H and ξ'_L are subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_L(\tau^*)$, then

$$-\beta[\alpha_{HH}\xi_H + \alpha_{HL}\xi_L](1+\rho) \leq -\beta[\alpha_{LH}\xi'_H + \alpha_{LL}\xi'_L](1+\rho);$$

and, (iv) for all $b > b^*_H(x^q_L(\mathbf{v}_0))$

$$\frac{1-\tau_L(b;\mathbf{v}_0)}{(1-\tau_L(b;\mathbf{v}_0)(1+\varepsilon))} > \frac{1-\tau_H(b;\mathbf{v}_0)}{(1-\tau_H(b;\mathbf{v}_0)(1+\varepsilon))}.$$

We will now establish that these four conditions are satisfied. For the first, we will show that $x^n_H(\mathbf{v}_0) \geq x^n_L(\mathbf{v}_0)$. Recall that by definition $x^n_\theta(\mathbf{v}_0)$ is the largest element in the compact set

$$X^n_\theta(\mathbf{v}_0) = \arg\max_x \{\frac{x}{n} + \beta\left[\alpha_{\theta H}\widetilde{v}_H(x) + \alpha_{\theta L}\widetilde{v}_L(x)\right] : x \in [x^*_\theta, \overline{x}]\}.$$

As shown in Lemma 2, we have that $x^*_H \geq x^*_L$. We can assume wlog that $x^n_L(\mathbf{v}_0) > x^*_L$. Thus, there exists sub-gradients ξ_H and ξ_L of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $x^n_L(\mathbf{v}_0)$ such that

$$\frac{1}{n} = -\beta\left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right].$$

Suppose that $x \leq x_L^n(\mathbf{v}_0)$. Then, if ξ_H' and ξ_L' of $\widetilde{v}_H$ and $\widetilde{v}_L$ at x then since $-\xi_H \leq -\xi_L$, $\alpha_{HH} \geq \alpha_{LH}$, and $-\xi_\theta' \leq -\xi_\theta$, we know that:

$$-\beta\left[\alpha_{HH}\xi_H' + \alpha_{HL}\xi_L'\right] \leq -\beta\left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right] = \frac{1}{n}.$$

This implies that $x_H^n(\mathbf{v}_0) \geq x_L^n(\mathbf{v}_0)$. A similar argument establishes the second condition.

For the third condition, let $b \in (b_H^*(x_H^n(\mathbf{v}_0)), b_L^*(x_L^q(\mathbf{v}_0)))$, let ξ_H and ξ_L be subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_H(\tau^*)$, and let ξ_H' and ξ_L' be subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $pg^* + (1+\rho)b - R_L(\tau^*)$. Then we have

$$\begin{aligned} -\beta[\alpha_{HH}\xi_H + \alpha_{HL}\xi_L](1+\rho) &\leq -\beta[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L](1+\rho) \\ &\leq -\beta[\alpha_{LH}\xi_H' + \alpha_{LL}\xi_L'](1+\rho), \end{aligned}$$

where the first inequality follows from the facts that $\alpha_{HH} \geq \alpha_{LH}$ and $-\xi_H \leq -\xi_L$, and the second inequality follows from the facts that $\widetilde{v}_H$ and $\widetilde{v}_L$ are concave and that $R_H(\tau^*) > R_L(\tau^*)$.

For the fourth condition, note that $(\tau_\theta(b;\mathbf{v}_0), g_\theta(b;\mathbf{v}_0), x_\theta(b;\mathbf{v}_0))$ is defined by the following three conditions:

$$nAg_\theta(b;\mathbf{v}_0)^{-\sigma} = p[\frac{1-\tau_\theta(b;\mathbf{v}_0)}{1-\tau_\theta(b;\mathbf{v}_0)(1+\varepsilon)}],$$

there exist subgradients ξ_H and ξ_L be subgradients of $\widetilde{v}_H$ and $\widetilde{v}_L$ at $x_\theta(b;\mathbf{v}_0)$ such that

$$[\frac{1-\tau_\theta(b;\mathbf{v}_0)}{1-\tau_\theta(b;\mathbf{v}_0)(1+\varepsilon)}] = -\beta n[\alpha_{\theta H}\xi_H + \alpha_{\theta L}\xi_L],$$

and

$$B_\theta(\tau_\theta(b;\mathbf{v}_0), g_\theta(b;\mathbf{v}_0), x_\theta(b;\mathbf{v}_0); b) = 0.$$

Suppose to the contrary that $\tau_H(b;\mathbf{v}_0) \geq \tau_L(b;\mathbf{v}_0)$. Then, $g_H(b;\mathbf{v}_0) \leq g_L(b;\mathbf{v}_0)$ and $x_H(b;\mathbf{v}_0) \geq x_L(b;\mathbf{v}_0)$. It follows that

$$\begin{aligned} 0 &= B_H(\tau_H(b;\mathbf{v}_0), g_H(b;\mathbf{v}_0), x_H(b;\mathbf{v}_0); b) \geq B_H(\tau_L(b;\mathbf{v}_0), g_L(b;\mathbf{v}_0), x_L(b;\mathbf{v}_0); b) \\ &> B_L(\tau_L(b;\mathbf{v}_0), g_L(b;\mathbf{v}_0), x_L(b;\mathbf{v}_0); b). \end{aligned}$$

This is a contradiction.

Now assume that the claim is true for $t = 1, ...k$ and consider it for $k+1$. By the induction step, for any $b \in [\underline{x}, \overline{x}]$ if ξ_{Lk} is a sub-gradient of v_{Lk} at b and ξ_{Hk} is a sub-gradient of v_{Hk} at b then we have that: $-\xi_{Lk} \geq -\xi_{Hk}$. It follows that v_{Hk} and v_{Lk} have the same properties as the functions $\widetilde{v}_H$ and $\widetilde{v}_L$ and the same argument as above applies to step $k+1$. ■

We can now prove Lemma A.3. Given Lemma 1, all we need to do is to establish that if $b \in (b_H^*, \overline{x}]$ then

$$-v_L'(b) \geq -v_H'(b).$$

Suppose, to the contrary, that there exists some $x \in (b_H^*, \overline{x}]$ such that $-v_L'(x) < -v_H'(x)$. Let $\varepsilon > 0$ be such that $x - \varepsilon > b_H^*$. Given that $\mathbf{v}_k$ converges to (v_H, v_L), it must be the case that $x_H^q(\mathbf{v}_k)$ converges to $x_H^q(v_H, v_L) = x_H^*$ as $k \to \infty$. Thus, for sufficiently large k, $b_H^*(x_H^q(\mathbf{v}_k)) < x - \varepsilon$. For any k sufficiently large, therefore, the Claim implies that v_{Hk} and v_{Lk} are differentiable on $(x - \varepsilon, \overline{x}]$ and $-v_{Lk}'(b) > -v_{Hk}'(b)$ for any $b \in (x - \varepsilon, \overline{x}]$. Thus, by Theorem 25.7 of Rockafellar (1970), we know that $\lim_{k\to\infty} v_{\theta k}'(b) = v_\theta'(b)$ for any $b \in (x - \varepsilon, \overline{x}]$, which includes x: a contradiction. ■

8.7 Proof of Lemma 3

(i) If $b \leq b_L^*$, we have that $x_L(b) = x_L^* > b_L^* \geq b$. Assume then that $b > b_L^*$. Suppose, contrary to the claim, that $x_L(b) \leq b$. By Lemma 1, we have that

$$-\beta n v_L'(b) = \frac{1 - \tau_L(b)}{1 - \tau_L(b)(1+\varepsilon)}.$$

Since $x_L(b) < \overline{x}$ the first order conditions for $(\tau_L(b), g_L(b), x_L(b))$ imply that there must exist sub-gradients ξ_L and ξ_H of v_L and v_H at $x_L(b)$ such that

$$\frac{1 - \tau_L(b)}{1 - \tau_L(b)(1+\varepsilon)} = -\beta n \left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right].$$

Since $\tau_L(b) > \tau^*$, for this equation to hold we must have that $x_L(b) > b_L^*$ and hence we know by Lemma 1 that

$$\xi_L = -\frac{1 - \tau_L(x_L(b))}{1 - \tau_L(x_L(b))(1+\varepsilon)}(\frac{1+\rho}{n}).$$

In addition, it must be the case that

$$-\beta\xi_H < -\beta\xi_L.$$

Clearly, this is case if $x_L(b) \leq b_H^*$. If $x_L(b) > b_H^*$, the inequality follows from the fact that $\tau_L(x_L(b)) > \tau_H(x_L(b))$. Thus, we have that

$$\begin{aligned} \frac{1 - \tau_L(b)}{1 - \tau_L(b)(1+\varepsilon)} &= -\beta n \left[\alpha_{LH}\xi_H + \alpha_{LL}\xi_L\right] \\ &< -\beta n \xi_L \\ &= \frac{1 - \tau_L(x_L(b))}{1 - \tau_L(x_L(b))(1+\varepsilon)}. \end{aligned}$$

But this is a contradiction because the facts that $\tau_L(\cdot)$ is increasing and that $b \geq x_L(b)$, imply that $\tau_L(b) \geq \tau_L(x_L(b))$.

(ii) If $b \leq b_H^*$, we have that $x_H(b) = x_H^*$. Thus, $x_H(b) > b$ if $b < x_H^*$ and $x_H(b) < b$ if $b \in (x_H^*, b_H^*]$. Assume then that $b > b_H^*$. Suppose, contrary to the claim, that $x_H(b) \geq b$. By Lemma 1, we have that

$$-\beta n v'_H(b) = \frac{1-\tau_H(b)}{1-\tau_H(b)(1+\varepsilon)}.$$

Since $x_H(b) \geq b > b_H^* > b_L^*$, then we know from the first order condition for $x_H(b)$ and Lemma 1 that

$$\frac{1-\tau_H(b)}{1-\tau_H(b)(1+\varepsilon)} \geq \alpha_{LH}\left(\frac{1-\tau_H(x_H(b))}{1-\tau_H(x_H(b))(1+\varepsilon)}\right) + \alpha_{LL}\left(\frac{1-\tau_L(x_H(b))}{1-\tau_L(x_H(b))(1+\varepsilon)}\right) \ (= \text{ if } x_H(b) < \overline{x}),$$

Since $\tau_H(x_H(b)) < \tau_L(x_H(b))$, this equation implies that

$$\frac{1-\tau_H(b)}{1-\tau_H(b)(1+\varepsilon)} > \left(\frac{1-\tau_H(x_H(b))}{1-\tau_H(x_H(b))(1+\varepsilon)}\right).$$

But this is a contradiction because the facts that $\tau_H(\cdot)$ is increasing and $b \leq x_H(b)$, imply that $\tau_H(b) \leq \tau_H(x_H(b))$. ■

8.8 Proof of Proposition 3

The dynamic pattern of debt described in the proposition follows immediately from Lemma 3. Thus, to prove the proposition we must show that the debt distribution converges strongly to a unique invariant distribution. To this end, define the state space $S = [\underline{x}, \overline{x}] \times \{L, H\}$ with associated σ-algebra $\mathcal{F} = \mathcal{B} \times \mathcal{H}$, where $\mathcal{B}$ is the family of Borel sets that are subsets of $[\underline{x}, \overline{x}]$, and $\mathcal{H}$ is the family of subsets of $\{L, H\}$. For any subset $A \in \mathcal{F}$, let $\mu_t(A)$ denote the probability that the state lies in A in period t. The probability measure μ_t describes the debt distribution in period t; for example, the probability that in period t the debt level lies between x_a and x_b in a boom is given by $\mu_t(([x, x], H))/\mu_t(([\underline{x}, \overline{x}], H))$. We are thus interested in the long run behavior of μ_t.

The probability distribution μ_1 depends on the initial level of debt b_0 and the initial state of the economy. To describe the probability distribution in periods $t \geq 2$ we must first describe the transition function implied by the equilibrium. This transition function is given by:

$$Q(A\,|b,\theta\,) = \begin{cases} \sum_{\{\theta':\ \text{s.t.}(x_{\theta'}(b),\theta') \in A\}} \alpha_{\theta\theta'} & \text{if } \exists\ \theta'\ \ s.t.\ \ (x_{\theta'}(b), \theta') \in A \\ 0 & otherwise \end{cases}.$$

Intuitively, $Q(A\,|b,\theta\,)$ is the probability that a set A is reached in one step if the initial state is (b,θ). Using this notation, the probability distribution in period $t \geq 2$ is defined inductively as:

$$\mu_t(A) = \sum_{\theta} \int_b Q(A\,|b,\theta\,)\mu_{t-1}(db,\theta).$$

The probability distribution μ^* is an invariant distribution if

$$\mu^*(A) = \sum_{\theta} \int_b Q(A\,|b,\theta\,)\mu^*(db,\theta).$$

We now show that the sequence of distributions $\langle \mu_t \rangle_{t=1}^{\infty}$ converges strongly to a unique invariant distribution.

By Theorem 11.12 in Stokey, Lucas and Prescott (1989), it is enough to show that the transition function Q satisfies the M *condition* (see the definition in Stokey, Lucas and Prescott (1989)). To this end, let $Q^1(A\,|b,\theta\,) = Q(A\,|b,\theta\,)$ and define recursively:

$$Q^n(A\,|b,\theta\,) = \sum_{\theta'} \int_{b'} Q(A\,|b',\theta'\,)Q^{n-1}(db',\theta'\,|b,\theta\,).$$

Thus, $Q^n(A\,|b,\theta\,)$ is the probability that a set A is reached in n steps if the initial state is (b,θ). To establish that Q satisfies the M condition, it is sufficient to prove that there exists a state (x^*,θ^*), an integer $N \geq 1$ and a number $\varepsilon > 0$, such that for any initial state (b,θ), $Q^N((x^*,\theta^*)\,|b,\theta\,) > \varepsilon$ (See Exercises 11.5 and 11.4 in Stokey, Lucas and Prescott (1989)).

Consider the state (x_H^*, H). Define $\xi = \min_{b \in [b_H^*, \overline{x}]} [b - x_H(b)]$. Since, by Lemma 3, $x_H(b) < b$ for any $b \in [b_H^*, \overline{x}]$, we have that $\xi > 0$. Let N be the smallest integer larger than $\frac{\overline{x} - b_H^*}{\xi} + 1$. Then, we claim that for any initial state (b,θ), we have that:

$$Q^N((x_H^*, H)\,|b,\theta\,) \geq \alpha_{LH} \left(\alpha_{HH}\right)^{N-1} > 0.$$

If this claim is true, then by choosing $\varepsilon \in \left(0, \alpha_{LH} \left(\alpha_{HH}\right)^{N-1}\right)$, we have the desired condition.

To see that the claim is true, suppose first that the initial state (b,θ) is such that $b \leq b_H^*$. With probability of at least α_{LH} the state will be (x_H^*, θ_H) in the next period and it will remain there for as long as the economy remains in a boom (which happens with probability α_{HH}). Next suppose that the initial state (b,θ) is such that $b > b_H^*$. With probability of at least α_{LH} the economy will be in a boom the next period and, again, it will remain in a boom thereafter with probability α_{HH}. If it does remain in a boom, then for as long as the debt level remains above b_H^*, debt will be reduced by at least ξ in each period. Thus, after N periods, the debt level must have gone below b_H^* in some period and therefore will have reached x_H^*. ■

8.9 Proof of Proposition 5

We proceed in two steps. First we prove that in a political equilibrium, the upperbound on debt $x \leq \overline{x}$ does not bind for any $b < \overline{x}$. This establishes equation (22) in Section 5.2. Then we prove the statement of Proposition 5.

Step 1. Consider a particular political equilibrium and let $\{\tau_\theta(b), g_\theta(b), x_\theta(b)\}$ be the associated equilibrium policies. We wish to prove that for any state (b, θ) there is an $\epsilon(b, \theta) > 0$ such that $x_\theta(b) < \overline{x} - \epsilon(b, \theta)$. Assume that there is a state (b, θ) such that $x_\theta(b)$ is arbitrarily close to $\overline{x}$; that is, $x_\theta(b) = \overline{x} - \zeta$, where ζ is arbitrarily small. We can write $g_\theta(b) = \phi(\tau_\theta(b))$ where $\phi(\tau)$ is a continuous function implicitly defined by the solution of the equation $nAg^{-\sigma} = [\frac{1-\tau}{1-\tau(1+\varepsilon)}]p$. Since $x_\theta(b) > x^*_\theta$, we must have

$$B_\theta(\tau_\theta(b), \phi(\tau_\theta(b)), x_\theta(b); b) = 0. \tag{32}$$

Thus, we can express all the policy choices as a function of ζ, where $x(b) = \overline{x} - \zeta = x(\zeta)$, $\tau_\theta(b) = \tau(\zeta)$ solves (32) and $g_\theta(b) = \phi(\tau(\zeta)) = g(\zeta)$. Note that as $\zeta \to 0$, we have $\tau(\zeta) \to \widetilde{\tau} < 1/(1+\varepsilon)$. For if $\tau(\zeta) \to 1/(1+\varepsilon)$, then $g(\zeta) \to 0$ and (32) would not be satisfied since $b < \overline{x}$. Moreover, $\tau(\zeta) \to \widetilde{\tau}$ implies $g(\zeta) \to \widetilde{g} > 0$.

From the first order condition on debt, we have that:

$$\begin{aligned} \left(\frac{1-\tau(\zeta)}{1-\tau(\zeta)(1+\varepsilon)}\right) &\geq -\beta[\alpha_{\theta H} v'_H(x(\zeta)) + \alpha_{\theta L} v'_L(x(\zeta))] \\ &\geq -\beta\alpha_{\theta L} v'_L(x(\zeta)) = -\beta\alpha_{\theta L}\left(\frac{1-\tau_L(x(\zeta))}{1-\tau_L(x(\zeta))(1+\varepsilon)}\right). \end{aligned} \tag{33}$$

It is easy to see that $\tau_L(x(\zeta)) \to 1/(1+\varepsilon)$ as $\zeta \to 0$. This implies that the right hand side of (33) diverges to infinity, while the left hand side converges to a finite value: a contradiction. ■

Step 2. We now prove that the deadweight loss of taxation is a submartingale when $q < n$, with strict inequality for some states (b, θ). Using Lemma 1, we can rewrite equation (22) as follows:

$$\frac{1-\tau_\theta(b)}{1-\tau_\theta(b)(1+\varepsilon)} = \Pr(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta) + \sum_{\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'}} \alpha_{\theta\theta'} \frac{1-\tau_{\theta'}(x_\theta(b))}{1-\tau_{\theta'}(x_\theta(b))(1+\varepsilon)}. \tag{34}$$

Now recall from the characterization that when $x_\theta(b)$ is less than or equal to $b^*_{\theta'}$, the tax rate $\tau_{\theta'}(x_\theta(b))$ will equal r^*. Thus, equation (34) can be rewritten as:

$$\frac{1-\tau_\theta(b)}{1-\tau_\theta(b)(1+\varepsilon)} = E\left[\frac{1-\tau_{\theta'}(x_\theta(b))}{1-\tau_{\theta'}(x_\theta(b))(1+\varepsilon)} \,|\theta\right] - \Pr(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} \,|\theta)\left[\frac{\varepsilon\tau^*}{1-\tau^*(1+\varepsilon)}\right]. \tag{35}$$

This implies that the MCPF is a submartingale (i.e., equation (35)). To complete the statement of the proposition, note that if $q < n$, then $\tau^* > 0$, and $b^*_\theta > \underline{x}$. It is also easy to show that if $q < n$, there is a $b' > x^*_H$ such that for any $b \leq b'$ we have $\Pr(\theta' \text{ s.t. } x_\theta(b) \leq b^*_{\theta'} | \theta) > 0$ for any θ. For these states (23) holds as a strict inequality. ■

8.10 Data Sources and Definitions

1. Output: seasonally adjusted real GDP, chained, base year is 2005. *Source: National Income and Product Accounts.*

   Available on-line at http://www.bea.gov/national/nipaweb/SelectTable.asp

2. The debt/GDP ratio: the end of the calendar year outstanding U.S. government debt not "held by Federal Government Accounts" as a % of GDP. *Source: Historical Tables of the office of Management and Budget, the White House.*

   Available on-line at http://www.whitehouse.gov/omb/budget/Historicals

3. The public spending/GDP ratio: total government expenditures less "Net Interest" as a % of GDP. *Source as above.*

4. The tax revenue/GDP ratio: the sum of "Individual Income Taxes", "Corporation Income Taxes", "Social Insurance and Retirement Receipts", "Excise", and "Other" as a % of GDP. *Source as above.*

5. Average labor income tax: as constructed by McDaniel (2007).

8.11 Additional Empirical Facts and Notes

Second Moments. Tables A1-4 extend our description of empirical facts in Section 6. They report additional measures of second moments of the fiscal variables. Table A1 confirms that debt is volatile, much more so than spending or the tax rate, and that these patterns do not depend on the measures used (for debt and public spending: the levels or as fractions of GDP; for the tax rate: the revenue/GDP ratio or the average labor income tax rate) or the detrending method.[35]

Table A2 shows that the correlation of debt and output is strong and negative regardless of the measure used or the detrending method. When HP-filtered, public spending is counter-cyclical

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[35] Given the relatively small length of our time period, we prefer the linear detrending to HP-filtering. Nevertheless, for completeness we report second moments computed using the HP-filtered data.

and the revenue/GDP ratio is procyclical. The correlations are statistically significant at the 1% level.[36] Table A3 indicates that all fiscal variables are persistent and that debt exhibits the most persistence. Finally, Table A.4 shows that these correlations are robust to the length of the time period over which they are computed.

	$\frac{\text{Debt}}{\text{GDP}}$	Debt	$\frac{\text{Spending}}{\text{GDP}}$	Spending	$\frac{\text{Revenue}}{\text{GDP}}$	Average Labor Tax Rate
Detrended	6.80	5.53^{+}	1.51	1.74^{+}	1.15	0.89
HP-filtered	3.17	3.34^{+}	0.86	0.77^{+}	0.91	0.67

Table A1. Empirical volatility of fiscal variables, *Extended.*[37]

	$\frac{\text{Debt}}{\text{GDP}}$	Debt	$\frac{\text{Spending}}{\text{GDP}}$	Spending	$\frac{\text{Revenue}}{\text{GDP}}$	Average Labor Tax Rate
Detrended	-0.89^{***}	-0.81^{***}	-0.18	0.10	0.26	-0.23
HP-filtered	-0.69^{***}	-0.61^{***}	-0.83^{**}	-0.60^{***}	0.49^{***}	0.30

Table A2. Empirical correlations of fiscal variables with output, *Extended.*[38]

	$\frac{\text{Debt}}{\text{GDP}}$	Debt	$\frac{\text{Spending}}{\text{GDP}}$	Spending	$\frac{\text{Revenue}}{\text{GDP}}$	Average Labor Tax Rate
Detrended	0.95^{***}	0.92^{***}	0.83^{***}	0.92^{***}	0.73^{***}	0.78^{***}
HP-filtered	0.79^{***}	0.80^{***}	0.59^{***}	0.65^{***}	0.64^{***}	0.64^{***}

Table A3. Empirical autocorrelations of fiscal variables, *Extended.*

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[36] This is the only case in which linear detrending and HP-filtering lead to significantly different results. To further investigate this issue, we band-pass filtered the data. The resulting correlations are weaker than with HP-filtering, -0.42 and 0.35, respectively and are significant at the 5% level and the 10% level, respectively.

[37] Variables marked by $^{+}$ are measured in percent of (average) GDP.

[38] The superscript *** is used to denote correlations that are statistically significant at the 1% level.

Time Period	$\frac{\text{Debt}}{\text{GDP}}$	Debt	$\frac{\text{Spending}}{\text{GDP}}$	Spending	$\frac{\text{Revenue}}{\text{GDP}}$
1950-2009	-0.57***	-0.54***	-0.49***	-0.47***	0.35***
1979-2009	-0.69***	-0.61***	-0.83***	-0.60***	0.49***

Table A4. Empirical correlations of HP-filtered variables with output over different time periods.

Recessions. We define recession years as the NBER recession years and first years immediately after the NBER recessions. This is because an NBER recession captures a "period from a peak to a trough,"[39] not the overall length of an economic downturn. As Figure 5 shows, the economy stays well below the trend after the recessions (as defined by NBER) end. In fact, after each NBER recession during our time period, detrended GDP does not improve in the subsequent year.

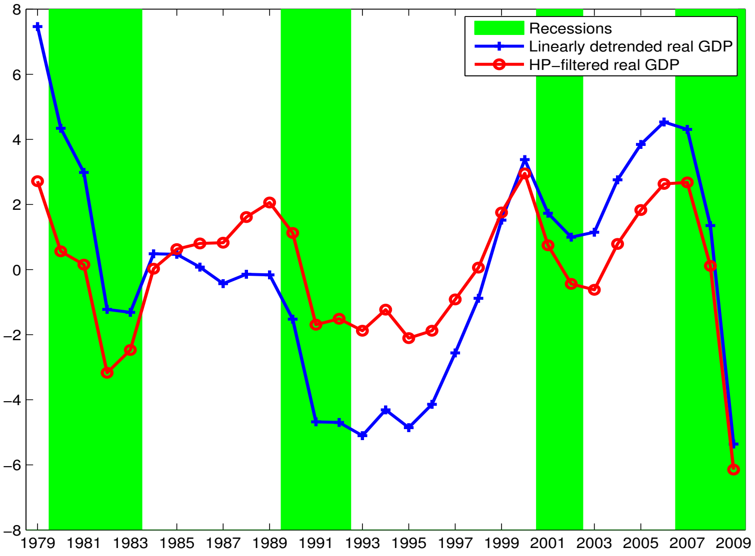

Figure 5: Recessions and GDP.

Average changes in debt and debt/GDP ratio during booms and recessions. These statistics are computed as the averages over the changes in debt (the debt/GDP ratio) during the

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[39] See http://www.nber.org/cycles/jan08bcdc_memo.html

booms (or recessions). Note that these are unconditional moments, in particular their weighted sum does not have to be zero.