Plan

1. Background: “Distributional Macroeconomics”

2. Overview of training school

3. Introduction to Hamilton-Jacobi-Bellman Equations
What do I mean by “Distributional Macroeconomics”?

• Study of macroeconomic questions in terms of distributions rather than just aggregates
  
  • typical example: distributions of income and wealth

• More technically: macroeconomic theories in which relevant state variable is a distribution (or: “heterogeneous agent models”)
Main Message

- Hard to coherently think about macro if ignore distribution

- Instead, rich interaction:

  distribution $\leftrightarrow$ macroeconomy

- Or perhaps more precisely:

  macroeconomy is a distribution
Inequality in Macro: A History of Thought

I find it useful to categorize macroeconomic theories as follows:

- **before modern macro**: 1930 to 1970
- **1st generation modern macro**: 1970 to 1990
- **2nd generation modern macro**: 1990 to financial crisis
- **3rd generation modern macro**: after the financial crisis

Main drivers of evolution in modern macro era

1. better data
2. better computers & algorithms
3. current events (rising inequality, financial crisis)
Before Modern Macro: 1930 to 1970

1. Keynesian IS/LM: about aggregates, no role for inequality/distribution by design

2. Distribution does play role in growth theory
   - mostly factor income distribution: Kaldor, Pasinetti and other Cambridge UK theorists
   - rarely personal income distribution: e.g. Stiglitz, Blinder

3.Disconnected empirical work on inequality (Kuznets)
First Generation Macro Theories: 1970 to 1990

Representative agent models, e.g. RBC & New Keynesian models

About aggregates, no role for inequality/distribution by design

Advertised as “microfounded” but representative agent assumption cuts 1st generation modern macro from much of micro research
First Generation Macro Theories: 1970 to 1990

What’s wrong with that?

1. cannot speak to a number of important empirical facts, e.g.
   • unequally distributed growth
   • poorest hit hardest in recessions

2. cannot think coherently about welfare – “who gains, who loses?”
Second generation theories incorporate heterogeneity from micro data, particularly in income and wealth.
Second generation theories represent economy with a distribution...
Second generation theories represent economy with a distribution... that moves over time, responding to macroeconomic shocks, policies
To contrast these theories with representative agent models, they are often referred to as “heterogeneous agent models.”

- important early contributions in the 1990s by Aiyagari, Bewley, Huggett, Krusell-Smith, Den Haan,...
Second generation theories can potentially speak to

- unequally distributed growth
- poorest hit hardest in recessions

and are useful for welfare analysis.
Second Generation Theories: Inequality $\not\Rightarrow$ Macro

- Typical finding: heterogeneity doesn’t matter much for macro agg’s
- Reason: in these theories, rich and poor differ in wealth but not consumption and saving behavior – rich = scaled version of poor
- Hence “inequality $\not\Rightarrow$ macro”, but also a knife-edge result
- Problem: in data, rich $\neq$ scaled version of poor, e.g. rich have
  - lower MPCs out of transitory income changes
  - higher saving rates out of permanent income, wealth
- Note: some important contributions from same time period don’t fit my narrative
  - Banerjee-Newman, Benabou, Galor-Zeira, Persson-Tabellini, ...
  - also related: 1950s “capitalist-worker theories” of Kaldor, Pasinetti, ...
Third Generation Theories: after the Crisis

• 3rd generation theories take micro data more seriously

• Leads them to emphasize things like
  • household balance sheets
  • credit constraints
  • MPCs that are high on average but heterogeneous
  • non-homotheticities, non-convexities

• Typical finding: distribution matters for macro

• Will see a number of examples throughout the course
Inequality in Macro: Summary

• **Before modern macro:** 1930 to 1970
  • it’s complicated

• **1st generation:** 1970 to 1990
  • representative agent models (RBC, New Keynesian etc)
  • no role for inequality by design

• **2nd generation:** 1990 to financial crisis
  • early “distributional macro” models
  • “macro ⇒ inequality” but “macro ≠ inequality”

• **3rd generation:** after the financial crisis
  • current “distributional macro” models
  • rich interaction: “inequality ↔ macro”
Inequality in Modern Macro: Summary


• “Prior to the financial crisis, representative-agent models were the dominant paradigm for analyzing many macroeconomic questions [= 1st generation].”

• “However, a disaggregated approach seems needed to understand some key aspects of the Great Recession...”

• “While the economics profession has long been aware that these issues matter, their effects had been incorporated into macro models only to a very limited extent prior to the financial crisis [= 2nd generation].”

• “I am glad to now see a greater emphasis on the possible macroeconomic consequences of heterogeneity [= 3rd generation].”
Distributional Macroeconomics: Summary

- Central banks and other policy institutions lack framework for thinking about distributional implications of macro policies
- Current macro research offers exactly that: economy = joint distribution of micro variables, not collection of aggregates

- Often: can’t ignore distribution even if care only about aggregates
- Not yet part of policy makers’ toolkit, but starting to change:
  - various central banks currently developing their own 3rd generation frameworks
Overview of Training School
What this training school is about

• Methods for solving “3rd generation” models with emphasis on monetary policy applications

• “Distributional macro” is hard
  • closed-form solutions are rare
  • computations are challenging

• Goal: teach you some new methods that make progress on this
  • solving heterogeneous agent model = solving PDEs
  • main difference to existing continuous-time literature:
    handle models for which closed-form solutions do not exist

• based on joint work with Yves Achdou, SeHyoun Ahn, Jiequn Han, Greg Kaplan, Pierre-Louis Lions, Jean-Michel Lasry, Gianluca Violante, Tom Winberry, Christian Wolf
Solving het. agent model = solving PDEs

• More precisely: a system of two PDEs
  1. Hamilton-Jacobi-Bellman equation for individual choices
  2. Kolmogorov Forward equation for evolution of distribution

• Many well-developed methods for analyzing and solving these
  • Codes: http://www.princeton.edu/~moll/HACTproject.htm

• Apparatus is very general: applies to any heterogeneous agent model with continuum of atomistic agents
  1. heterogeneous households (Aiyagari, Bewley, Huggett,...)
  2. heterogeneous producers (Hopenhayn,...)

• can be extended to handle aggregate shocks (Krusell-Smith,...)
Computational Advantages relative to Discrete Time

1. **Borrowing constraints only show up in boundary conditions**
   - FOCs always hold with “=”

2. **“Tomorrow is today”**
   - FOCs are “static”, compute by hand: \( c^{-\gamma} = v_a(a, y) \)

3. **Sparsity**
   - solving Bellman, distribution = inverting matrix
   - but matrices very sparse (“tridiagonal”)
   - reason: continuous time ⇒ one step left or one step right

4. **Two birds with one stone**
   - tight link between solving (HJB) and (KF) for distribution
   - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
   - reason: diff. operator in (KF) is adjoint of operator in (HJB)
Real Payoff: extends to more general setups

- non-convexities
- stopping time problems (no need for threshold rules)
- multiple assets
- aggregate shocks
What you’ll be able to do at end of the training school

- Joint distribution of income and wealth in Aiyagari model

![3D graph showing the joint distribution of income and wealth. The x-axis represents income (z), the y-axis represents wealth (a), and the z-axis represents density (g(a,z,t)). The graph is a color-coded surface that peaks at the origin and decreases as you move away from it, indicating a high density of observations near the origin of income and wealth.](image)
What you’ll be able to do at end of the training school

- Experiment: effect of one-time redistribution of wealth
What you’ll be able to do at end of the training school

Video of convergence back to steady state

https://www.dropbox.com/s/op5u2n1ifmmer2o/distribution_tax.mp4?dl=0
Admin

• Schedule: see https://cepr.org/40016

• Automatic differentiation

  • as stated on course website, you should install SeHyoun Ahn’s automatic differentiation package
    https://github.com/sehyoun/MATLABAutoDiff

  • will need this for perturbation method stuff on Day 3

  • apparently some problems with compilation of .mex files, will see if we can resolve these after today’s Tutorial
Introduction to HJB Equations
Outline

1. Now: Hamilton-Jacobi-Bellman equations in deterministic settings

2. Next lecture
   - numerical solution using finite difference method
   - HJB equations in stochastic settings
Hamilton-Jacobi-Bellman Equation: Some “History”

• Aside: why called “dynamic programming”?

• Bellman: “Try thinking of some combination that will possibly give it a pejorative meaning. It’s impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.”

Hamilton-Jacobi-Bellman Equations

• Pretty much all deterministic optimal control problems in continuous time can be written as

\[ v(x_0) = \max_{\{\alpha(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} r(x(t), \alpha(t)) \, dt \]

subject to the law of motion for the state

\[ \dot{x}(t) = f(x(t), \alpha(t)) \quad \text{and} \quad \alpha(t) \in A \]

for \( t \geq 0, \ x(0) = x_0 \) given.

• \( \rho \geq 0 \): discount rate
• \( x \in X \subseteq \mathbb{R}^N \): state vector
• \( \alpha \in A \subseteq \mathbb{R}^M \): control vector
• \( r : X \times A \rightarrow \mathbb{R} \): instantaneous return function
Example: Neoclassical Growth Model

\[ v(k_0) = \max_{\{c(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} u(c(t)) \, dt \]

subject to

\[ \dot{k}(t) = F(k(t)) - \delta k(t) - c(t) \]

for \( t \geq 0, \) \( k(0) = k_0 \) given.

- Here the state is \( x = k \) and the control \( \alpha = c \)
- \( r(x, \alpha) = u(\alpha) \)
- \( f(x, \alpha) = F(x) - \delta x - \alpha \)
• How to analyze these optimal control problems? Here: “cookbook approach”

• **Result**: the value function of the generic optimal control problem satisfies the Hamilton-Jacobi-Bellman equation

\[ \rho v(x) = \max_{\alpha \in A} r(x, \alpha) + v'(x) \cdot f(x, \alpha) \]

• In the case with more than one state variable \( N > 1 \), \( v'(x) \in \mathbb{R}^N \) is the gradient of the value function.
Example: Neoclassical Growth Model

- “cookbook” implies:

\[ \rho v(k) = \max_c u(c) + v'(k)(F(k) - \delta k - c) \]

- Proceed by taking first-order conditions etc

\[ u'(c) = v'(k) \]

- Tutorial: derivation from discrete time Bellman equation
Some general, somewhat philosophical thoughts

• MAT 101 way ("first-order ODE needs one boundary condition") is not the right way to think about HJB equations

• these equations have very special structure which one should exploit when analyzing and solving them

• Particularly true for computations

• Important: all results/algorithms apply to problems with more than one state variable, i.e. it doesn’t matter whether you solve ODEs or PDEs
Existence and Uniqueness of Solutions to (HJB)

Recall Hamilton-Jacobi-Bellman equation:

$$\rho v(x) = \max_{\alpha \in A} \left\{ r(x, \alpha) + v'(x) \cdot f(x, \alpha) \right\}$$  \hspace{1cm} (HJB)

Two key results, analogous to discrete time:

• **Theorem 1** (HJB) has a unique “nice” solution

• **Theorem 2** “nice” solution equals value function, i.e. solution to “sequence problem”

• Here: “nice” solution = “viscosity solution”

• See supplement “Viscosity Solutions for Dummies”
  http://www.princeton.edu/~moll/viscosity_slides.pdf

• Theorems 1 and 2 hold for both ODE and PDE cases, i.e. also with multiple state variables...

• ... also hold if value function has kinks (e.g. from non-convexities)

• Remark re Thm 1: in typical application, only very weak boundary conditions needed for uniqueness (≤’s, boundedness assumption)
References: Some “Third Generation” Papers

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References: Other Academic Articles

• Aiyagari (1994) “Uninsured Idiosyncratic Risk and Aggregate Saving”


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• Persson & Tabellini (1994) “Is Inequality Harmful for Growth?”
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  http://microeconomicinsights.org/wealthy-hand-to-mouth-households-key-to-understanding-the-impacts-of-fiscal-stimulus/


- Yellen (2016) “Macroeconomic Research After the Crisis”
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