Outline

• Last time: New Keynesian 3 equation model, derived from micro foundations

• Ignored ZLB (or “liquidity trap”), $i(t) \geq 0$.

• This time: optimal policy at ZLB?

• Also optimal fiscal policy.
Advantages of Continuous Time

- Very nice and tractable
- Payoff: very clean results, even though complicated stuff.
  - Keep interest rate at zero past liquidity trap
  - Engineer output boom, not inflation
  - Lots more
- Policy involves optimal switching time, awkward in discrete time
- Graphical analysis using phase diagrams
- “Aerospace engineering approach” to optimal monetary policy: central bank controls trajectory of economy.
Model

• Last time: three equation model

\[
\begin{align*}
\dot{x} &= i - \pi - r \quad \text{(IS')} \\
\dot{\pi} &= \rho\pi - \kappa x \quad \text{(PC')} \\
i &= i^* + \phi\pi + \phi_x x \quad \text{(TR')} 
\end{align*}
\]

• Recall: \( \kappa = (\varepsilon - 1)(1 + \varphi)/\theta \) = price flexibility

• This time: drop Taylor rule (TR’), replace with optimal monetary policy

• Also generalize to CRRA utility, \( \sigma \neq 1 \), impose ZLB

\[
\begin{align*}
\dot{x} &= \sigma^{-1}(i - \pi - r) \\
\dot{\pi} &= \rho\pi - \kappa x \\
i &\geq 0
\end{align*}
\]
The Natural Interest Rate

- Have shown last time: if \( r(t) > 0 \) for all \( t \)
  - first-best \( (x(t), \pi(t)) = (0, 0) \) can be attained, e.g. with Taylor rule with \( i^* = r \) and \( \phi > 1 \).
  - “Divine coincidence”

- This time: liquidity trap scenario:

\[
 r(t) = \begin{cases} 
 r, & t \in [0, T) \\
 \bar{r}, & t \in [T, \infty) 
\end{cases}
\]

where \( r < 0 < \bar{r} \).

- Why could natural interest rate go negative?
  - TFP growth down \( r = \rho + \dot{A}/A \).
  - Anything that affects savings behavior \( \rho \) (“animal spirits”,…)
  - Credit crunch (Guerrieri and Lorenzoni, 2011)
Liquidity Trap: No Commitment

- “No commitment” means central bank benevolent but cannot credibly announce plans for the future.
- Acts opportunistically at each point in time.
- Will see momentarily: this is a bad thing
- After the trap, \( t \in [T, \infty) \): implement first best
  \[
  (x(t), \pi(t)) = (0, 0)
  \]
- How do this? See last lecture. For example, Taylor rule with \( i^* = r, \phi > 1 \).
Liquidity Trap: No Commitment

- During the trap, \( t \in [0, T) \) : \( i(t) = 0 \), cannot attain first-best
- Dynamics governed by
  \[
  \dot{x} = -\sigma^{-1}(\xi + \pi) \\
  \dot{\pi} = \rho \pi - \kappa x
  \]
- Important: terminal condition
  \[
  (x(T), \pi(T)) = (0, 0)
  \]
No Commitment

- at $T$: first best $x(T) = \pi(T) = 0$
- before $T$: binding zero interest $i(t) = 0$
No Commitment

- Deflation and Depression...

**Proposition.**

**Without commitment**

\[ x(t) < 0, \pi(t) < 0 \text{ for } t < T \]

As \( T \to \infty \)

\[ x(0), \pi(0) \to -\infty \]

- intuition
  - **too** high real interest: too high growth
  - cumulative effect with \( T \)
Does Price Flexibility Help? No!

- Recall: \( \kappa = (\varepsilon - 1)(1 + \varphi)/\theta \)
- \( \theta \): scales price adjustment cost function
- **Proposition:** Higher \( \kappa \) (lower \( \theta \)) leads to lower \( x(t) \) and \( \pi(t) \).
  
  In the limit as \( \kappa \to \infty \) (\( \theta \to 0 \)) we have
  
  \[ x(t), \pi(t) \to -\infty \]

- Intuition: Phillips curve in integral form
  
  \[ \pi(t) = \kappa \int_{t}^{\infty} e^{-\rho(s-t)}x(s)ds \]

- For given **negative** \( x(s), s \geq t, \kappa \uparrow \Rightarrow \pi \downarrow \), i.e. more deflation.
- From Euler equation (IS curve)
  
  \[ \dot{x} = -\sigma^{-1}(r + \pi) \uparrow \]

- \( \Rightarrow x(0) \downarrow \) since \( x(T) = 0 \) fixed.
- Discontinuity? No. \( \pi(T) = 0 \) suboptimal with \( \kappa = \infty \).
Elbow Room

- No commitment \( \Rightarrow \) deflation, depression
- Even simple, non-optimal policies make things better.
- Here’s one. **For all** \( t \geq 0 \) **also after trap ends**, \( t > T \), set

\[
\pi(t) = -r > 0, \quad x(t) = -\frac{1}{\kappa}r > 0
\]

- Problem: real interest rate too high.
- Partial fix: inflation forever
- Small positive output gap
- Can check \( i(t) \geq 0 \) for all \( t \).
- Key: be able to commit to something other than

\[
(\pi(T), x(T)) = (0, 0)
\]
Optimal Monetary Policy with Commitment

• Planning problem

$$\min_{c,\pi,i} \frac{1}{2} \int_0^{\infty} e^{-\rho t} \left( x(t)^2 + \lambda \pi(t)^2 \right) dt$$

$$\dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho \pi(t) - \kappa x(t)$$

$$i(t) \geq 0$$

and $x(0), \pi(0)$ free.

• Objective function: welfare loss, can be derived as second order approximation to welfare around zero inflation.

See Gali (2008), Chapter 4, Appendix 1; Woodford (2003) Proposition 6.4.

• Note: high $x$ bad because work too much ($MRS \neq MPL$)
Optimal Monetary Policy with Commitment

- Hamiltonian:

\[ \mathcal{H} = \frac{1}{2} x^2 + \frac{1}{2} \lambda \pi^2 + \mu_x \sigma^{-1} (i - r - \pi) + \mu_\pi (\rho \pi - \kappa x) - \psi i \]

- \((x, \pi)\): states (output gap, inflation)
- \((\mu_x, \mu_\pi)\): co-states
- \(i\): control (nominal interest rate)
- \(\psi \geq 0\): Lagrange mult. on \(i \geq 0\), compl. slackness \(\psi i = 0\).

- Conditions for optimum:

\[ \mu_x \sigma^{-1} = \psi \implies \mu_x \geq 0, \mu_x i = 0 \]

\[ \dot{\mu}_x = \rho \mu_x - x + \kappa \mu_\pi \]

\[ \dot{\mu}_\pi = \rho \mu_\pi - \lambda \pi + \sigma^{-1} \mu_x - \rho \mu_\pi \]
Optimal Monetary Policy with Commitment

- Massage a bit

\[
\begin{align*}
\mu_x & \geq 0, \quad \mu_x i = 0 \\
\dot{\mu}_x &= \rho \mu_x - x + \kappa \mu_\pi \\
\dot{\mu}_\pi &= -\lambda \pi + \sigma^{-1} \mu_x \\
\dot{x} &= \sigma^{-1} (i - \pi - r) \\
\dot{\pi} &= \rho \pi - \kappa x
\end{align*}
\]

- Since \(x(0), \pi(0)\) are free, two additional conditions

\[
\mu_x(0) = 0, \quad \mu_\pi(0) = 0
\]

- Recall: \(\mu_x(0) = \) marginal value of one additional unit of \(x(0)\) and similarly for \(\pi(0)\).

- + two transversality conditions.
Graphical Representation

- Three Phases:
  - Phase I: During the Liquidity Trap, \( t \in [0, T) \)
  - Phase II: Just out of the Trap, \( t \in [T, \hat{T}) \)
  - Phase III: After the Storm, \( t \in [\hat{T}, \infty) \)

- Go backwards in time. Draw phase diagrams III, II, I.

- Phase diagrams will be such that there is a unique \((x(0), \pi(0))\) that satisfies transversality conditions.

- By picking a time path for the nominal interest rate, \( i(t) \), the central bank can pick these initial conditions and the trajectories for \((x(t), \pi(t)), t > 0\).

- Trick is sticking together three phase diagrams in right way.

- Note: it's all about whether \((x(t), \pi(t))\) are fixed or free at \( t = 0, T, \hat{T} \).
Phase III: After the Storm

• \((\pi(\hat{T}), x(\hat{T}))\) inherited from the past, i.e. not free.

• First-best \((\pi(t), x(t)) = (0, 0)\) generally not feasible

• ZLB not binding: \(\mu_x = \dot{\mu}_x = 0.\)

\[
\begin{align*}
\dot{\mu}_x &= \rho \mu_x - x + \kappa \mu_\pi \\
\dot{\mu}_\pi &= -\lambda \pi + \sigma^{-1} \mu_x
\end{align*}
\Rightarrow
\begin{align*}
x &= \kappa \mu_\pi \\
\dot{\mu}_\pi &= -\lambda \pi
\end{align*}
\Rightarrow
\begin{align*}
\dot{x} &= \kappa \dot{\mu}_\pi = -\kappa \lambda \pi
\end{align*}

• Combining with \(\dot{x} = \sigma^{-1}(i - \pi - r)\)

\[i = r + (1 - \kappa \sigma \lambda) \pi \equiv l(r, \pi)\]

• Same interest rate condition as in Clarida, Gali and Gertler (1999). Property: \(\pi = 0 \Rightarrow i = l(r, 0) = r.\)
Phase III: After the Storm

- System with optimal control $i = I(\pi, r) = r + (1 - \kappa \sigma \lambda)\pi$

$$\dot{x} = -\kappa \lambda \pi$$

$$\dot{\pi} = \rho \pi - \kappa x$$

- Draw phase diagram $\Rightarrow$ saddle path

- **Claim**: In phase III (when the ZLB is slack)

$$x(t) = \phi \pi(t), \quad \phi \equiv \frac{\rho + \sqrt{\rho^2 + 4\lambda \kappa^2}}{2\kappa}$$

- That is, $\phi$ is **slope of the saddle path**.
Ignoring lower bound

\[ i(t) = I(\pi(t), r(t)) \]

\[ x = \phi \pi \]
Phase III: After the Storm

- **Claim:** Saddle path is

\[ x(t) = \phi \pi(t), \quad \phi \equiv \frac{\rho + \sqrt{\rho^2 + 4\lambda \kappa^2}}{2\kappa} \]

- Derivation (more general trick for finding slope of saddle path)
- With optimal control \( i = I(r, \pi) \)

\[ \frac{dx}{dt} = -\kappa \lambda \pi \]

\[ \frac{d\pi}{dt} = \rho \pi - \kappa x \]

- Saddle path: \( x(\pi(t)) \) such that this holds. Slope:

\[ \frac{dx}{d\pi} = \frac{dx/dt}{d\pi/dt} = \frac{-\kappa \lambda \pi}{\rho \pi - \kappa x(\pi)} \]

- Guess \( x(\pi) = \phi \pi \) (doesn’t always work, if not use L’Hopital)

\[ \phi = \frac{-\kappa \lambda}{\rho - \kappa \phi} \quad \Rightarrow \quad -\kappa \phi^2 + \phi \rho + \kappa \lambda = 0 \]

- Solve quadratic, two roots \( \phi \), take positive one.
Phase II: Just out of the Trap

- Recall: $i(t) = 0$ in this phase
- Again $(\pi(T), x(T))$ inherited from the past, i.e. not free.
- System

\[
\begin{align*}
\dot{x} &= -\sigma^{-1}(\bar{r} + \pi) \\
\dot{\pi} &= \rho\pi - \kappa x
\end{align*}
\]

- Same phase diagram as no-commitment case except that $\dot{x} = 0$ locus at $\pi = -\bar{r}$ rather than $\pi = -\bar{r}$
\[ \dot{x} = 0 \]
\[ \dot{\pi} = 0 \]

Lower bound binding

\[ x = \phi \pi \]

\[ i(t) = 0 \]
Phase I: During the Liquidity Trap

• System

\[ \dot{x} = -\sigma^{-1}(r + \pi) \]
\[ \dot{\pi} = \rho\pi - \kappa x \]

• Same phase diagram as no-commitment case.

• \((x(0), \pi(0))\) free, but \((x(T), \pi(T))\) given.
During trap

\[ x(t) = \phi \pi \]

\[ \dot{x} = 0 \]

\[ \dot{\pi} = 0 \]

\[ i(t) = 0 \]
Putting it all together!

\[ \dot{\pi} = 0 \]

\[ x = \phi \pi \]
commitment

pure discretion
Main Results

\[ I(\pi, r) \equiv r + (1 - \kappa \sigma \lambda)\pi \]

- result 1: slack ZLB \( i(t) = I(\pi(t), r(t)) \)
- result 2:
  \[ I(\pi(t), r(t)) < 0 \text{ for } t \in [t_0, t_1) \]
  \[ i(t) = 0 \text{ for } t \in [t_0, t_2) \text{ with } t_1 < t_2 \]
- result 3: inflation must be positive at some point
- result 4: output takes both signs
- result 5: inflation may be positive throughout
Communication

- What kind of commitment?
  - **needed**: policy commitments for t>T...
    - 1. promised targets
      \[ x(T), \pi(T) \]
    - 2. interest rate and exit inflation \( \hat{T} > T \)
      \[ i(t) = 0 \text{ for } t < \hat{T} \]
      \[ \pi(\hat{T}) \]
  - **irrelevant**: policy commitments for t<T
Inflation or Boom?

- Literature: purpose of monetary policy in liquidity trap = promote inflation.

- Werning: not true, real objective = engineer consumption boom.

- Paper: three special cases
  - Special case 1: if process fully rigid so that inflation is zero, still want to set $i(t) = 0$ past trap.
  - Special case 3: arbitrary no inflation constraint, $\pi \leq 0$. 

#2 Rigid Prices

- **completely rigid prices**

\[
x(t) = \sigma^{-1} \int_{t}^{\hat{T}} r(s) ds
\]

- \( \hat{T} = T \) ➞ \( x(t) < 0 \) (no commitment)
- \( \hat{T} > T \) ➞ \( \uparrow x(t) \) and \( x(T) > 0 \)
Inflation or Boom?

- Want to set
  \[ \int_{0}^{\infty} e^{-\rho t} x(t) dt = 0 \]
- Current recession and subsequent boom should average out.
- Intuition: lower future interest rates to discourage savings
Avoiding Inflation

\[ x = \phi \pi \]
Government Spending

• Utility $U(C, N, G)$

• Public goods, $G$, valued in $U$, resources: $C + G = Y$.

• Planning problem

$$\min_{c, \pi, i, g} \frac{1}{2} \int_0^\infty e^{-\rho t} \left( (c(t) + (1 - \Gamma)g(t))^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt$$

$$\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho \pi(t) - \kappa(c(t) + (1 - \Gamma)g(t))$$

$$i(t) \geq 0$$

and $x(0), \pi(0)$ free.

• $c = (C - C^*)/C^* \approx \log C - \log C^*$, $g = (G - G^*)/C^*$

• $\Gamma$ is neoclassical multiplier

• Flexible prices: optimal spending $c = -(1 - \Gamma)g$
Spending

- Gap $x = c + (1 - \Gamma)g$ transformation

$$\min_{x, \pi, i, g} \frac{1}{2} \int_0^\infty e^{-\rho t} \left( x(t)^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt$$

$$\dot{x}(t) = (1 - \Gamma)\dot{g}(t) + \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho \pi(t) - \kappa x(t)$$

$$i(t) \geq 0$$

$$x(0), \pi(0) \text{ free.}$$

- spending loosens Euler equation

**Proposition.** Spending is initially positive. But falls over time, and becomes negative.

- front-loading
Stimulus

- Decomposition
  - “opportunistic”: static cost-benefit...
    \[ g^*(c) \equiv \arg \max_g U(c, c + g, g) \]
  - low c \rightarrow low (shadow) wage \rightarrow higher g

- “stimulus”...
  \[ \hat{g}(t) = g(t) - g^*(c(t)) \]
  - attempt to manipulate consumption
Opportunistic vs. Stimulus Spending

- Decomposition
  - Total spending (blue)
  - Opportunistic spending (green)
  - Stimulus spending (red)
Stimulus

- Planning Problem

\[
\min_{\hat{x}, \pi, i, \hat{g}} \frac{1}{2} \int_0^\infty e^{-\rho t} \left( c(t)^2 + \hat{\lambda}\pi(t)^2 + \hat{\eta}\hat{g}(t)^2 \right) dt
\]

\[
\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))
\]

\[
\dot{\pi}(t) = \rho\pi(t) - \kappa (\psi c(t) + (1 - \Gamma)\hat{g}(t))
\]

\[
i(t) \geq 0,
\]

\[
c(0), \pi(0) \text{ free.}
\]

- stimulus: loosens Phillips Curve
Proposition. Stimulus:
(a) initially zero;
(b) may be zero throughout;
(c) switches signs: starts positive, then negative, or vice versa
Optimal Fiscal Policy: Summary

- Almost all spending opportunistic, stimulus = very small component
- Opportunistic spending does affect private consumption, by affecting inflation. “Leaning against the wind” mitigates both deflations and inflations.
- However, effects are incidental, would have been obtained by completely myopic policy maker.
- Model **not** screaming for stimulus
Mixed Commitment

- Monetary: discretionary
- Spending: Commitment up to $t = T$
mixed commitment
Conclusions

- **Liquidity trap**
  - no commitment: deflation and depression
  - worse with flexible prices

- **Monetary policy**
  - avoids deflation
  - commitment important

- **Fiscal policy**
  - countercyclical
  - all opportunistic

- **Mixed commitment**
  - role for extra stimulus
  - larger if prices are more flexible
Related Literature