Outline

(1) Hamilton-Jacobi-Bellman equations in deterministic settings (with derivation)

(2) Numerical solution: finite difference method

(3) Stochastic differential equations
Aside: why called “dynamic programming”?

Bellman: “Try thinking of some combination that will possibly give it a pejorative meaning. It’s impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.”

http://www.ingre.unimore.it/or/corsi/vecchi_corsi/complementiromaterieledidattico/originidp.pdf
Hamilton-Jacobi-Bellman Equations

- Recall the generic deterministic optimal control problem from Lecture 1:

\[ V(x_0) = \max_{u(t)_{t=0}^\infty} \int_0^\infty e^{-\rho t} h(x(t), u(t)) \, dt \]

subject to the law of motion for the state

\[ \dot{x}(t) = g(x(t), u(t)) \text{ and } u(t) \in U \]

for \( t \geq 0, \ x(0) = x_0 \) given.

- \( \rho \geq 0 \): discount rate
- \( x \in X \subseteq \mathbb{R}^m \): state vector
- \( u \in U \subseteq \mathbb{R}^n \): control vector
- \( h : X \times U \rightarrow \mathbb{R} \): instantaneous return function
Example: Neoclassical Growth Model

\[ V(k_0) = \max_{c(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} U(c(t)) dt \]

subject to

\[ \dot{k}(t) = F(k(t)) - \delta k(t) - c(t) \]

for \( t \geq 0, \ k(0) = k_0 \) given.

- Here the state is \( x = k \) and the control \( u = c \)
- \( h(x, u) = U(u) \)
- \( g(x, u) = F(x) - \delta x - u \)
Generic HJB Equation

- The value function of the generic optimal control problem satisfies the Hamilton-Jacobi-Bellman equation

\[ \rho V(x) = \max_{u \in U} h(x, u) + V'(x) \cdot g(x, u) \]

- In the case with more than one state variable \( m > 1 \), \( V'(x) \in \mathbb{R}^m \) is the gradient of the value function.
Example: Neoclassical Growth Model

- “cookbook” implies:

\[ \rho V(k) = \max_c U(c) + V'(k)[F(k) - \delta k - c] \]

- Proceed by taking first-order conditions etc

\[ U'(c) = V'(k) \]
Derivation from Discrete-time Bellman

- Here: derivation for neoclassical growth model.
- Extra class notes: generic derivation.
- Time periods of length $\Delta$
- discount factor

$$\beta(\Delta) = e^{-\rho\Delta}$$

- Note that $\lim_{\Delta \to 0} \beta(\Delta) = 1$ and $\lim_{\Delta \to \infty} \beta(\Delta) = 0$.
- Discrete-time Bellman equation:

$$V(k_t) = \max_{c_t} \Delta U(c_t) + e^{-\rho\Delta} V(k_{t+\Delta}) \quad \text{s.t.}$$

$$k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$
Derivation from Discrete-time Bellman

- For small $\Delta$ (will take $\Delta \to 0$), $e^{-\rho \Delta} = 1 - \rho \Delta$

$$V(k_t) = \max_{c_t} \Delta U(c_t) + (1 - \rho \Delta)V(k_{t+\Delta})$$

- Subtract $(1 - \rho \Delta)V(k_t)$ from both sides

$$\rho \Delta V(k_t) = \max_{c_t} \Delta U(c_t) + (1 - \Delta \rho)[V(k_{t+\Delta}) - V(k_t)]$$

- Divide by $\Delta$ and manipulate last term

$$\rho V(k_t) = \max_{c_t} U(c_t) + (1 - \Delta \rho)\frac{V(k_{t+\Delta}) - V(k_t)}{k_{t+\Delta} - k_t} \frac{k_{t+\Delta} - k_t}{\Delta}$$

Take $\Delta \to 0$

$$\rho V(k_t) = \max_{c_t} U(c_t) + V'(k_t)\dot{k}_t$$
Connection Between HJB Equation and Hamiltonian

- Hamiltonian
  \[ \mathcal{H}(x, u, \lambda) = h(x, u) + \lambda g(x, u) \]

- Bellman
  \[ \rho V(x) = \max_{u \in U} h(x, u) + V'(x)g(x, u) \]

- Connection: \( \lambda(t) = V'(x(t)) \), i.e. co-state = shadow value

- Bellman can be written as
  \[ \rho V(x) = \max_{u \in U} \mathcal{H}(x, u, V'(x)) \]

- Hence the “Hamilton” in Hamilton-Jacobi-Bellman

- Can show: playing around with FOC and envelope condition gives conditions for optimum from Lecture 1.
Numerical Solution: Finite Difference Method

- Example: Neoclassical Growth Model

\[ \rho V(k) = \max_c U(c) + V'(k)[F(k) - \delta k - c] \]

- Functional forms

\[ U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad F(k) = k^\alpha \]

- See material at

  http://www.princeton.edu/~moll/HACTproject.htm

particularly

  - Code 1: http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m
  - Code 2: http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m
Diffusion Processes

- A diffusion is simply a continuous-time Markov process (with continuous sample paths, i.e. no jumps)
- Simplest possible diffusion: standard Brownian motion (sometimes also called “Wiener process”)
- **Definition:** a standard Brownian motion is a stochastic process \( W \) which satisfies

\[
W(t + \Delta t) - W(t) = \varepsilon_t \sqrt{\Delta t}, \quad \varepsilon_t \sim N(0, 1), \quad W(0) = 0
\]

- Not hard to see

\[
W(t) \sim N(0, t)
\]

- Continuous time analogue of a discrete time random walk:

\[
W_{t+1} = W_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, 1)
\]
Standard Brownian Motion

- Note: mean zero, $\mathbb{E}(W(t)) = 0$...
- ... but blows up $\text{Var}(W(t)) = t$. 
Brownian Motion

- Can be generalized
  \[ x(t) = x(0) + \mu t + \sigma W(t) \]

- Since \( \mathbb{E}(W(t)) = 0 \) and \( \text{Var}(W(t)) = t \)
  \[ \mathbb{E}[x(t) - x(0)] = \mu t, \quad \text{Var}[x(t) - x(0)] = \sigma^2 t \]

- This is called a Brownian motion with drift \( \mu \) and variance \( \sigma^2 \)

- Can write this in differential form as
  \[ dx(t) = \mu dt + \sigma dW(t) \]

  where \( dW(t) \equiv \lim_{\Delta t \rightarrow 0} \varepsilon_t \sqrt{\Delta t} \), with \( \varepsilon_t \sim N(0, 1) \)

- This is called a \textbf{stochastic differential equation}

- Analogue of stochastic difference equation:
  \[ x_{t+1} = \mu t + x_t + \sigma \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \]
Brownian Motion with Drift

"x(t)" : Current3
Forecast x : Current3
"Forecast x + 1 SD" : Current3
"Forecast x - 1 SD" : Current3
Further Generalizations: Diffusion Processes

- Can be generalized further (suppressing dependence of $x$ and $W$ on $t$)
  \[ dx = \mu(x)dt + \sigma(x)dW \]

  where $\mu$ and $\sigma$ are any non-linear etc etc functions.

- This is called a “diffusion process”

- $\mu(\cdot)$ is called the drift and $\sigma(\cdot)$ the diffusion.

- all results can be extended to the case where they depend on $t$, $\mu(x, t)$, $\sigma(x, t)$ but abstract from this for now.

- The amazing thing about diffusion processes: by choosing functions $\mu$ and $\sigma$, you can get pretty much any stochastic process you want (except jumps)
Example 1: Ornstein-Uhlenbeck Process

- Brownian motion $dx = \mu dt + \sigma dW$ is not stationary (random walk). But the following process is

\[ dx = \theta(\bar{x} - x)dt + \sigma dW \]

- Analogue of AR(1) process, autocorrelation $e^{-\theta} \approx 1 - \theta$

\[ x_{t+1} = \theta \bar{x} + (1 - \theta)x_t + \sigma \varepsilon_t \]

- That is, we just choose

\[ \mu(x) = \theta(\bar{x} - x) \]

and we get a nice stationary process!

- This is called an “Ornstein-Uhlenbeck process”
Ornstein-Uhlenbeck Process

Can show: stationary distribution is $N(\bar{x}, \sigma^2/(2\theta))$
Example 2: “Moll Process”

- Design a process that stays in the interval $[0, 1]$ and mean-reverts around $1/2$

  $$\mu(x) = \theta \left( \frac{1}{2} - x \right), \quad \sigma(x) = \sigma x (1 - x)$$

  $$dx = \theta \left( \frac{1}{2} - x \right) dt + \sigma x (1 - x) dW$$

- Note: diffusion goes to zero at boundaries $\sigma(0) = \sigma(1) = 0$ & mean-reverts $\Rightarrow$ always stay in $[0, 1]$
Other Examples

- Geometric Brownian motion:
  \[ dx = \mu x dt + \sigma x dW \]
  \[ x \in [0, \infty), \text{ no stationary distribution:} \]
  \[ \log x(t) \sim N((\mu - \sigma^2/2)t, \sigma^2 t). \]

- Feller square root process (finance: “Cox-Ingersoll-Ross”) \]
  \[ dx = \theta(\bar{x} - x) dt + \sigma \sqrt{x} dW \]
  \[ x \in [0, \infty), \text{ stationary distribution is } Gamma(\gamma, 1/\beta), \text{ i.e.} \]
  \[ f_\infty(x) \propto e^{-\beta x} x^{\gamma-1}, \quad \beta = 2\theta \bar{x}/\sigma^2, \quad \gamma = 2\theta \bar{x}/\sigma^2 \]

Next Time

1. Hamilton-Jacobi-Bellman equations in stochastic settings (without derivation)

2. Ito’s Lemma

3. Kolmogorov Forward Equations

4. Application: Power laws (Gabaix, 2009)