Consumption and Saving with an Indivisible Durable

1 Model Description

Individuals have flow utility over non-durable consumption $c_t$ and durable consumption $d_t$

$$u(c_t) + \kappa d_t.$$  

Durable consumption is indivisible: $d_t \in \{0, 1\}$. For instance, $d_t$ could present car ownership: individuals either own a car, $d_t = 1$, or they do not, $d_t = 0$. For concreteness we will therefore refer to the durable as “car.” Individuals who do not own a car can purchase it at price $p_0$. Individuals who already own a car can sell it at price $p_1$ with $p_1 < p_0$. At any time when they do not buy or sell a car, individuals’ wealth $a_t$ accumulates according to $\dot{a}_t = y + ra_t - c_t$ where $y$ is their constant labor income and $r$ is the interest rate. If they buy a car their wealth jumps down by $p_0$, and if they sell their car their wealth jumps up by $p_1$.

Denote by $v_d(a)$ the value of having wealth $a$ and car ownership state $d \in \{0, 1\}$. Individuals in state $d = 0$, optimally choose consumption and the stopping time $\tau$ at which to purchase the car:

$$v_0(a) = \max_{\{c_t\}_{t \geq 0, \tau}} \int_0^\tau e^{-\rho t} u(c_t) dt + e^{-\rho \tau} v_0^*(a_\tau)$$

$$\dot{a}_t = y + ra_t - c_t, \quad a_t \geq a, \quad a_0 = a.$$  

where $v_0^*(a)$ is the value of buying a car given by:

$$v_0^*(a) = \begin{cases} v_1(a - p_0), & \text{if } a - p_0 \geq a \\ -\infty, & \text{if } a - p_0 < a \end{cases}$$  

The second branch takes care of the borrowing constraint: individuals cannot buy a car if doing so would lead them to violate the borrowing constraint. The problem for individuals already owning a car is symmetric:

$$v_1(a) = \max_{\{c_t\}_{t \geq 0, \tau}} \int_0^\tau e^{-\rho t} (u(c_t) + \kappa) dt + e^{-\rho \tau} v_1^*(a_\tau)$$

$$\dot{a}_t = y + ra_t - c_t, \quad a_t \geq a, \quad a_0 = a.$$  

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1We thank Victor Rios-Rull for suggesting this Problem
where \( v^*_1(a) \) is the value of selling a car given by:

\[
v^*_1(a) = v_0(a + p_1)
\]

Because we will solve the problem on a bounded grid \( a \leq a \leq a_{\text{max}} \), we will make the simplifying assumption that \( v^*_1(a) = v_0(\max\{a+p_1, a_{\text{max}}\}) \), i.e. if selling the car would take the individual’s wealth above \( a_{\text{max}} \) then she receives a smaller price.

The individual’s problem boils down to a system of “HJB Variational Inequalities” (HJBVIs)

\[
0 = \min \{ \rho v_0(a) - \max_c \{ u(c) + v'_0(a)(y + ra - c) \} , v_0(a) - v^*_0(a) \} \\
0 = \min \{ \rho v_1(a) - \max_c \{ u(c) + \kappa + v'_1(a)(y + ra - c) \} , v_1(a) - v^*_1(a) \}
\]

See http://www.princeton.edu/~moll/HACTproject/option_simple.pdf for an explanation of HJBVIs.

2 Algorithm

The Matlab code at http://www.princeton.edu/~moll/HACTproject/car.m solves the system (3) and (4) under the assumption of CRRA utility \( u'(c) = c^{-\gamma} \). It uses a similar algorithm as in http://www.princeton.edu/~moll/HACTproject/option_simple.pdf and http://www.princeton.edu/~moll/HACTproject/option_simple.m.

A sketch is as follows: the discretized HJBVIs are basically:

\[
0 = \min \{ \rho v_0 - u(v_0) - A(v_0)v_0, v_0 - v^*_0(v_1) \}, \quad (3) \\
0 = \min \{ \rho v_1 - u(v_1) - A(v_1)v_1, v_1 - v^*_1(v_0) \} \quad (4)
\]

These can then be converted into a Linear Complementarity Problem (LCP) that can be solved with readily available solvers.

3 Results

Figure 1 plots the value and policy functions. As is intuitive, poor individuals sell their cars and rich individuals buy a car. The policy functions in panels (c) and (d) are only plotted for wealth levels at which individuals neither buy or sell a car, i.e. for which the indicators in panel (c) equal zero. The policy functions at other wealth levels are irrelevant because individuals immediately jump from state \( d = 0 \) to \( d = 1 \) or vice versa.
Figure 1: Value and Policy Functions